Penn
UNIVERSITY of PENNSYLVANIA

Math 371 in-class exam

December 11, 2017

No books, paper or any electronic device may be used. Please turn off your phones, and put all electronic devices, including calculators and phones, in your bags.

This examination consists of eight questions plus one extra credit problem. Partial credits will be given for *substantial* work.

Please **show all your work**.

NAME (PRINTED):

INSTRUCTOR: C.-L. Chai

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

1	2	3	4	5	6	7	8	9	10	Total

1. (10 pts) Give an example of a non-zero liner operator T on a vector space such that $T^3 = 0$ but $T^2 \neq 0$.

2. (10 pts) Give an example of a square matrix A with coefficients in $\mathbb{R}$ such that there exists an element $C \in \mathrm{GL}_n(\mathbb{C})$ such that $C \cdot A \cdot C^{-1}$ is a diagonal matrix, but there is no element $D \in \mathrm{GL}_n(\mathbb{R})$ such that $D \cdot A \cdot D^{-1}$ is a diagonal matrix.

3. (10 pts) Give an example of a non-noetherian commutative ring.
(Recall that a commutative ring is noetherian if every ideal is finitely generated. You need to show that the ring in your example is indeed not noetherian.)

4. (15 pts) Give (a) an exmaple of a commutative ring with exactly one prime ideal.

(b) Give an exmaple of a commutative ring with exactly two prime ideals.

(c) Give an example of a commutative ring with exactly three prime ideals.

5. (15 pts) Give an example of a unique factorization domain with is not a principle ideal domain.

(b) Give an example of a commutative local ring which is not a principle ideal domain.

6. (20 pts) (a) Find an element $A \in \mathrm{M}_2(\mathbb{Q})$ such that $A^2 + A + I_2 = 0$, where I_2 is the 2×2 identity matrix.

(b) Does there exist a ring homomorphism from $\mathbb{Q}[\sqrt{-3}] \cong \mathbb{Q}[x]/(x^2 + x + 1)$ to the matrix algebra $\mathrm{M}_2(\mathbb{Q})$? Exhibit one explicitly if there is one, otherwise prove that no such homomorphism exists.

7. (20 pts) Let T be a linear operator on a 10-dimensional vector space V over $\mathbb{R}$ such that the minimal polynomial of T is $x^3(x^2 - x + 4)^2$. Classify, up to conjugtion by elements of $\mathrm{GL}_{\mathbb{R}}(V)$, all linear operators T satisfying the above properties.
(Please provide a list of matrix representations satisfying the above properties, such that every matrix representation of such a linear operator T is conjugate to exactly on one your list.)

8. (15 extra pts) Prove that there are infinitely many prime ideals in the polynomial ring $\mathbb{F}_3[x]$.

9. (15 extra pts) (a) Give an example of a finite group which is not isomorphic to any subgroup of $\mathrm{GL}_3(\mathbb{Q})$.
(This is easier than part (b) below.)

(b) Give an example of a finite group which is not isomorphic to any subgroup of $\mathrm{GL}_3(\mathbb{C})$.

10. (20 extra pts) (a) Determine all ideals of the complex group algebra $\mathbb{C}[S_3]$ of the symmetric group S_3.

(b) Determine whether the number of left ideals of $\mathbb{C}[S_3]$ is finite or infinite. Give a complete list of these left ideal by providing an explicit description of them. (If there are infinitely many left ideals, your explicit description should come with a natural/explicit parametrization of them.)

(Hint for both parts: Recall that a left ideal of $\mathbb{C}[S_3]$ is a sub-representation of the left regular representation of S_3. The basic theory of linear representations tells you that the left regular representation of S_3 decomposes into the direct sum of irreducible sub-representations.)