Math 241 (Spring 2019) Homework 1 Solutions

January 31, 2019

Problem 1.2.3. The equation we want to derive is

$$c\rho A(x)\frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\left(A(x)\frac{\partial^2 u}{\partial x^2} + \frac{\partial A}{\partial x}\frac{\partial u}{\partial x}\right).$$

The expressions c, ρ, $A(x)$, $u(x,t)$, K_0 are defined as before. To get the above expression, the same argument used to derive equation (1.2.6) can be applied to yield

$$e(x,t) = c\rho u(x,t). \tag{1}$$

We now need to derive the conservation of heat energy with variable cross-sectional area $A(x)$. The analogous expression for equation (1.2.1) is

$$\frac{\partial}{\partial t}\left(e(x,t)A(x)\Delta x\right) \approx \phi(x,t)A(x) - \phi(x+\Delta x,t)A(x+\delta x).$$

(Note that in this problem $Q = 0$.) By dividing throughout by Δx, letting $x \to 0$, and using the product rule, the conservation law we get is

$$A(x)\frac{\partial e}{\partial t} = -\left(A(x)\frac{\partial \phi}{\partial x} + \phi(x,t)\frac{\partial A}{\partial x}\right). \tag{2}$$

To put everything together, we recall Fourier's law of heat conduction, which says that

$$\phi(x,t) = -K_0\frac{\partial u}{\partial x}. \tag{3}$$

Substituting equations (1) and (3) into equation (2) yields

$$c\rho A(x)\frac{\partial u}{\partial t} = A(x)\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + K_0\frac{\partial u}{\partial x}\frac{\partial A}{\partial x},$$

giving us the desired equation after dividing throughout by $c\rho$.

Problem 1.2.4. (a) The total amount of chemical in between x and Δx is given by $u(x,t)A\Delta x$. Looking at its time derivative,

$$\frac{\partial}{\partial t}\left[u(x,t)A\Delta x\right] \approx \phi(x,t)A - \phi(x+\Delta x,t)A$$
$$\frac{\partial u}{\partial t} \approx \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}$$

Take the limit as $\Delta x \to 0$:

$$\frac{\partial u}{\partial t} = \lim_{\Delta x\to 0}\frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}$$
$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$$

Then, use Fick's Law to get:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-k\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

(b) The total amount of chemical between $x = a$ and $x = b$ is given by $\int_a^b u(x,t)A\,dx$. Look at its time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)A\,dx = \phi(a,t)A - \phi(b,t)A$$

$$\int_a^b \frac{\partial u}{\partial t}\,dx = \phi(a,t) - \phi(b,t)$$

$$\int_a^b \frac{\partial u}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x}\right)\,dx = 0$$

Since this is valid for all a and b, the integrand must be 0 everywhere, so

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

Then, by substituting in Fick's Law once again we get the diffusion equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$.

Problem 1.2.8. The explanation before equation (1.2.6) tells us that the total thermal energy is

$$c\rho\int_0^L A(x)u(x,t)\,dx. \tag{4}$$

Problem 1.4.1. In case it is not clear, the relevant equation for this problem is

$$\frac{d^2u}{dx^2} + \frac{Q}{K_0} = 0,$$

gotten by letting $\partial u/\partial t = 0$ in equation (1.2.9). This is now a simple ordinary differential equation in one variable, so we just state the answers here.

(b) One should get

$$u(x) = T - \frac{T}{L}x$$

(c) One should get

$$u(x) = T.$$

(f) One should get

$$u(x) = -\frac{1}{12}x^4 + \frac{L^3}{3}x + T.$$

(g) One should get

$$u(x) = T - \frac{T}{1+L}x.$$

Problem 1.4.4. The total thermal energy in the rod is given by

$$\int_0^L c\rho u\,dx$$

To show that it is constant in time, we can show that its time derivative is zero.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L c\rho u \, dx = \int_0^L \frac{\partial}{\partial t}\left(c\rho u\right) \, dx$$
$$= \int_0^L c\rho \frac{\partial u}{\partial t} \, dx$$

We know from the heat equation that $\frac{\partial u}{\partial t} = \frac{K_0}{c\rho}\frac{\partial^2 u}{\partial x^2}$, so

$$= \int_0^L K_0 \frac{\partial^2 u}{\partial x^2} \, dx$$

This can just be integrated directly to give

$$= K_0 \frac{\partial u}{\partial x}\bigg|_0^L$$
$$= K_0 \left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right]$$

But since both ends of the rod are insulated, $\frac{\partial u}{\partial x}(L,t) = \frac{\partial u}{\partial x}(0,t) = 0$, so this is zero. Since the derivative of the total thermal energy is always zero, the total thermal energy remains constant.

Problem 1.4.7. (a) In steady state, $\frac{\partial u}{\partial t} = 0$, so we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 1 = 0$$
$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -1$$

This can be integrated twice to give the solution

$$u(x) = -\frac{x^2}{2} + C_1 x + C_2$$

To apply our boundary conditions, we need $\frac{\mathrm{d}u}{\mathrm{d}x}$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -x + C_1$$
$$\frac{\mathrm{d}u}{\mathrm{d}x}(0) = C_1 = 1$$
$$\frac{\mathrm{d}u}{\mathrm{d}x}(L) = -L + C_1 = \beta$$

$C_1 = 1$ and $\beta = -L + 1$ will solve this system of equations. So, we first know that there is only a solution when $\beta = -L + 1$, and that

$$u(x) = -\frac{x^2}{2} + x + C_2$$

To find C_2, we need to look at the total thermal energy; note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L c\rho u \, dx = \int_0^L c\rho \left(\frac{\partial^2 u}{\partial x^2} + 1\right) \, dx$$
$$= c\rho \left(\frac{\partial u}{\partial x} + x\right)\bigg|_0^L$$
$$= c\rho \left(\beta - 1 + L\right) = 0$$

So, the initial and steady-state thermal energies must be the same:

$$\int_0^L u(x)\,dx = \int_0^L f(x)\,dx$$

$$-\frac{L^3}{6} + \frac{L^2}{2} + C_2 L = \int_0^L f(x)\,dx$$

$$C_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2} + \frac{L^3}{6}\right)$$

(b) Follow similar steps to part (a). This time, the equilibrium equation is $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0$, so the solution is

$$u(x) = C_1 x + C_2$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = C_1$$

Applying the boundary conditions, we get that $C_1 = 1$, and the requirement that $\beta = 1$. Again, the thermal energy is constant in time, so

$$\int_0^L u(x)\,dx = \int_0^L f(x)\,dx$$

$$\frac{L^2}{2} + C_2 L = \int_0^L f(x)\,dx$$

$$C_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^2}{2}\right)$$

(c) Now we get $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \beta - x$, which has solution

$$u(x) = -\frac{x^3}{6} + \beta\frac{x^2}{2} + C_1 x + C_2$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{x^2}{2} + \beta x + C_1$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(0) = C_1 = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(L) = -\frac{L^2}{2} + \beta L = 0$$

So, we only have an equilibrium solution when $\beta = \frac{L}{2}$, and

$$u(x) = -\frac{x^3}{6} + \beta\frac{x^2}{2} + C_2$$

The thermal energy is constant in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L c\rho u\,dx = \int_0^L c\rho\left(\frac{\partial^2 u}{\partial x^2} + x - \beta\right)\,dx$$

$$= c\rho\left(\frac{L^2}{2} - \beta L\right) = c\rho\left(\frac{L^2}{2} - \frac{L^2}{2}\right) = 0$$

So, we can say

$$\int_0^L u(x)\,dx = \int_0^L f(x)\,dx$$

$$-\frac{L^4}{24} + \frac{L^4}{12} + C_2 L = \int_0^L f(x)\,dx$$

$$C_2 = \frac{1}{L}\left(\int_0^L f(x)\,dx - \frac{L^4}{24}\right)$$

Problem 1.4.10. .In this problem the cross-sectional area is constant, so $A(x) = A$. if we compare our data with equation (1.2.9), one sees that $c\rho = K_0$. Now, note that

$$u(x,T) = \int_0^T \frac{\partial u}{\partial t}\, dt + u(x,0)$$

Thus equation (4) of Problem 1.2.8 tells us the total thermal energy at time $t = T$ is

$$\begin{aligned}
K_0 A \int_0^L u(x,T)\, dx &= K_0 A \int_0^T \int_0^L \frac{\partial u}{\partial t}\, dx\, dt + K_0 A \int_0^L u(x,0)\, dx \\
&= K_0 A \int_0^T \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + 4 \right) dx\, dt + K_0 A \int_0^L f(x)\, dx \\
&= K_0 A \int_0^T \left(\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) \right) dt + 4K_0 ALT + K_0 A \int_0^L f(x)\, dx \\
&= K_0 A \int_0^T (6-5)\ dt + 4K_0 ALT + K_0 A \int_0^L f(x)\, dx \\
&= K_0 AT + 4K_0 ALT + K_0 A \int_0^L f(x)\, dx.
\end{aligned}$$

Problem 1.4.11. (a) The total thermal energy is given by $\int_0^L c\rho u\, dx$. To get it as a function of time, look at its time derivative:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\text{Energy}) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L c\rho u\, dx \\
&= c\rho \int_0^L \frac{\partial u}{\partial t}\, dx \\
&= c\rho \int_0^L \left(\frac{\partial^2 u}{\partial x^2} + x \right) dx \\
&= c\rho \left[\frac{\partial u}{\partial x} + \frac{x^2}{2} \right] \Bigg|_0^L \\
&= c\rho \left[\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t) + \frac{L^2}{2} \right] \\
&= c\rho \left(7 - \beta + \frac{L^2}{2} \right)
\end{aligned}$$

Integrate this to get the energy:

$$\begin{aligned}
\text{Energy} &= c\rho t \left(7 - \beta + \frac{L^2}{2} \right) + \text{Energy}(t=0) \\
&= c\rho t \left(7 - \beta + \frac{L^2}{2} \right) + \int_0^L c\rho f(x)\, dx
\end{aligned}$$

(b) An equilibrium exists only if the rate of change of the total thermal energy is 0: this happens if $\beta = 7 + \frac{L^2}{2}$. We can then use this to find the steady state temperature distribution. In equilibrium, $\frac{\partial u}{\partial t} = 0$, so we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + x = 0$$

which has the solution

$$u(x) = -\frac{x^3}{6} + C_1 x + C_2$$

for which

$$\frac{\partial u}{\partial x} = -\frac{x^2}{2} + C_1$$

$$\frac{\partial u}{\partial x}(0) = C_1 = \beta$$

$$\frac{\partial u}{\partial x}(L) = -\frac{L^2}{2} + \beta = -\frac{L^2}{2} + 7 + \frac{L^2}{2} = 7$$

So,

$$u(x) = -\frac{x^3}{6} + \beta x + C_2$$

We can use the total thermal energy to figure out the value of C_2.

$$\int_0^L c\rho u(x)\,dx = \int_0^L c\rho f(x)\,dx$$

$$\int_0^L \left[-\frac{x^3}{6} + \beta x + C_2\right] dx = \int_0^L f(x)\,dx$$

$$-\frac{L^4}{24} + \frac{\beta L^2}{2} + C_2 L = \int_0^L f(x)\,dx$$

$$C_2 = \frac{1}{L}\left[\int_0^L f(x)\,dx + \frac{L^4}{24} - \frac{\beta L^2}{2}\right]$$

Math 241 (Spring 2019) Homework 2 Solutions

February 7, 2019

Problem 2.2.2. (a) Clearly $L(u+cv) = L(u) + cL(v)$ for any constant c.

(b) In this case L will not be a linear operator if K_0 is not dependent on u. An example is $K_0(x,u) = u$.

Problem 2.2.4. (a) We are probably assuming linearity of L here. If so, then it is clear by the definition of a linear operator.

(b) An answer is $u_{p1} + u_{p2}$. (You can also add homogeneous solutions.)

Problem 2.3.1. The idea is to write $u(x,t) = \phi(x)G(t)$ and separate the stuff with ϕ and G after substituting this into the PDE. We give the answers below. (Your answers can vary up to signs and scaling.)

(b) One should get

$$G_t = -\lambda G, \qquad k\phi_{xx} - v_0\phi_x = -\lambda\phi.$$

(c) One should get

$$\phi_{xx} = -\lambda\phi, \qquad h_{yy} = \lambda h.$$

For this part the variables are x and y.

(d) One should get

$$G_t = -\lambda G, \qquad \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -\lambda\phi.$$

For this part the variables are r and t. One can also further simplify the second ODE using the chai rule.

(e) One should get

$$h_t = \lambda k h, \qquad \phi_{xxxx} = \lambda\phi.$$

Problem 2.3.2. The idea is to follow the same steps as outlined in section 2.3 of the book. You should consider the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. We give the answers below.

(a) One should get

$$\lambda = n^2, \;\; n = 1, 2, 3, \ldots.$$

(d) One should get

$$\lambda = \left(\frac{n\pi - \frac{\pi}{2}}{L}\right)^2, \;\; n = 1, 2, 3, \ldots.$$

Problem 2.3.3. (b) Since $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$ is already expressed as a linear combination of $\sin\frac{n\pi x}{L}$, we can proceed by matching coefficents to the Fourier series: $B_1 = 3$ and $B_3 = -1$. Then, using the general solution we found for this problem,

$$\boxed{u(x,t) = 3\sin\frac{\pi x}{L}e^{-k(\pi/L)^2 t} - \sin 3\pi x L e^{-k(3\pi/L)^2 t}}$$

(c) We just need to find the coefficients B_n for this initial condition.

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\ &= \frac{4}{L}\int_0^L \cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx \end{aligned}$$

Using the trig identity that $\sin\alpha\cos\beta = 1/2(\sin(\alpha+\beta) + \sin(\alpha-\beta))$, we get

$$
\begin{aligned}
&= \frac{2}{L}\int_0^L \left(\sin\frac{(n+3)\pi x}{L} + \sin\frac{(n-3)\pi x}{L}\right) dx \\
&= -\frac{2}{L}\left[\frac{L}{(n+3)\pi}\cos\frac{(n+3)\pi x}{L} + \frac{L}{(n-3)\pi}\cos\frac{(n-3)\pi x}{L}\right]\Bigg|_0^L
\end{aligned}
$$

If $n+3$ is even (i.e. n is odd), then these integrate to zero. However, if n is even, the cosine terms integrate to 2, and we get

$$
\begin{aligned}
B_n &= -\frac{2}{L}\left[\frac{2L}{(n+3)\pi} + \frac{2L}{(n-3)\pi}\right] \\
&= -\frac{4}{\pi}\left(\frac{2n}{n^2-9}\right) \\
&= -\frac{8n}{\pi(n^2-9)}
\end{aligned}
$$

So, our final answer is

$$
\boxed{u(x,t) = -\frac{8}{\pi}\sum_{n=2,4,6,\ldots}\left[\left(\frac{n}{n^2-9}\right)\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right]}
$$

(d)

$$
\begin{aligned}
B_n &= \frac{2}{L}\int_0^{L/2}\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L} 2\sin\frac{n\pi x}{L}\,dx \\
&= -\frac{2}{n\pi}\left(\cos\frac{n\pi x}{L}\right)\Bigg|_0^{L/2} - \frac{4}{n\pi}\left(\cos\frac{n\pi x}{L}\right)\Bigg|_{L/2}^{L} \\
&= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi} - \frac{4}{n\pi}\cos n\pi + \frac{4}{n\pi}\cos\frac{n\pi}{2} \\
&= \frac{2}{n\pi}\left[1 + \cos\frac{n\pi}{2} - 2\cos n\pi\right]
\end{aligned}
$$

There are three cases:

$$
B_n = \begin{cases} \frac{6}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is a multiple of 4} \\ -\frac{4}{n\pi} & \text{if } n \text{ is a multiple of 2 but not 4} \end{cases}
$$

With these values of B_n,

$$
u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}
$$

Problem 2.3.4. (a) The total thermal energy looks like:

$$
\begin{aligned}
\text{Energy} &= c\rho A\int_0^L u(x,t)\,dx \\
&= c\rho A\int_0^L \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\,dx \\
&= c\rho A\sum_{n=1}^{\infty} -\left[\frac{B_n L}{n\pi}\cos\frac{n\pi x}{L}\right]\Bigg|_0^L e^{-k(n\pi/L)^2 t} \\
&= c\rho A\sum_{\text{odd } n}\frac{2B_n L}{n\pi}e^{-k(n\pi/L)^2 t}
\end{aligned}
$$

The B_n's are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Problem 2.3.6.

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right]dx$$

As long as $n - m \neq 0$ and $n + m \neq 0$, we get

$$\begin{aligned} &= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L} + \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\right]\Bigg|_0^L \\ &= 0 \end{aligned}$$

If $n - m = 0$, but n and m are not 0, then the second term integrates to zero, but the first term is

$$\frac{1}{2}\int_0^L 1\,dx = \frac{L}{2}$$

If $n = m = 0$, then both terms become 1, and we get

$$\frac{1}{2}\int_0^L 2\,dx = L$$

So,

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0 \end{cases}$$

Problem 2.3.8. (a) In steady-state, we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{\alpha}{k}u$$

The solution for this is

$$u(x) = C_1\cosh\sqrt{\frac{\alpha}{k}}x + C_2\sinh\sqrt{\frac{\alpha}{k}}x$$

Plugging in that $u(0,t) = 0$, we get that $C_1 = 0$; plugging in $u(L,t) = 0$, we get

$$u(L) = C_2\sinh\sqrt{\frac{\alpha}{k}}L = 0$$

Since $\sinh x$ is never 0 except when $x = 0$, this means that $C_2 = 0$ as well, so the steady-state solution is just the trivial solution $u(x) = 0$.

(b) Solve this by separating variables: guess a solution of the form

$$u(x,t) = \phi(x)G(t)$$

The PDE now becomes:

$$\phi G' = k\phi'' G - \alpha\phi G$$

Divide through by $k\phi G$ to get

$$\frac{G'}{kG} = \frac{\phi''}{\phi} - \frac{\alpha}{k}$$

Since the left side is a function of t, and the right side is a function of x, they must be equal to a constant:

$$\frac{G'}{kG} = \frac{\phi''}{\phi} - \frac{\alpha}{k} = -\lambda$$

Solving the time portion is the same as usual:

$$G(t) = e^{-k\lambda t}$$

For the spatial part, we have

$$\phi''(x) = -\left(\lambda - \frac{\alpha}{k}\right)\phi(x)$$

With $\mu = \lambda - \alpha/k$

$$\phi''(x) = -\mu\phi(x)$$

with boundary conditions

$$\phi(0) = \phi(L) = 0$$

This is the same problem we solved in section 2.3; we know what the eigenvalues and eigenfunctions are:

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \ ; \ n = 1, 2, 3, ...$$

$$\phi_n(x) = \sin\frac{n\pi x}{L} \ ; \ n = 1, 2, 3, ...$$

For each value of μ_n we get a corresponding λ_n:

$$\lambda_n = \mu_n + \frac{\alpha}{k} = \left(\frac{n\pi}{L}\right)^2 + \frac{\alpha}{k}$$

Combining our solutions, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left((n\pi/L)^2 + \alpha/k\right)t}$$

To find the B_n's, we use the fact that

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$$

And so,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

In the limit that $t \to \infty$, all the exponential term in the series go to zero, and so the entire function goes to $u(x) = 0$, which is what we expected from part (a).

Math 241 (Spring 2019) Homework 3 Solutions

February 14, 2019

Problem 2.4.1 (a). From Section 2.4.1 of the book, the answer is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt},$$

where A_n is as specified in Equations (2.4.23) and (2.4.24). A computation tells us that

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L u(x,0)\,dx = \frac{1}{2}, \\ A_n &= \frac{2}{L}\int_0^L u(x,0)\cos\frac{m\pi x}{L}\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2} \qquad \text{for } n \geq 1. \end{aligned}$$

Problem 2.4.3. Write $s = -\lambda$. If $s > 0$ then the general solution is

$$\phi(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}.$$

The boundary conditions tells us that

$$\begin{aligned} c_1 + c_2 &= c_1 e^{2\sqrt{s}\pi} + c_2 e^{-2\sqrt{s}\pi}, \\ c_1 - c_2 &= c_1 e^{2\sqrt{s}\pi} - c_2 e^{-2\sqrt{s}\pi}. \end{aligned}$$

Thus $1 = e^{2\sqrt{s}\pi}$, a contradiction.

If $s = 0$ then the general solution is

$$\phi(x) = c_1 + c_2 x.$$

The boundary conditions tells us that

$$\begin{aligned} c_1 &= c_1 + 2c_2\pi, \\ c_2 &= c_2. \end{aligned}$$

Thus constant functions are eigenfunctions for $\lambda = 0$.

If $s < 0$ then the general solution is

$$\phi(x) = c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right).$$

The boundary conditions tells us that

$$\begin{aligned} c_2 &= c_1 \sin\left(2\sqrt{\lambda}\pi\right) + c_2 \cos\left(2\sqrt{\lambda}\pi\right), \\ c_1 &= c_1 \cos\left(2\sqrt{\lambda}\pi\right) + c_2 \sin\left(2\sqrt{\lambda}\pi\right). \end{aligned}$$

Solving for $\sin\left(2\sqrt{\lambda}\pi\right)$ and $\cos\left(2\sqrt{\lambda}\pi\right)$ gives us

$$\sin\left(2\sqrt{\lambda}\pi\right) = 0, \qquad \cos\left(2\sqrt{\lambda}\pi\right) = 1.$$

Hence we require $2\sqrt{\lambda}\pi = n\pi$ for $n \geq 1$, with corresponding eigenfunctions $\sin\left(\sqrt{\lambda}x\right)$ and $\cos\left(\sqrt{\lambda}x\right)$.

Summarizing, the eigenvalues and eigenfunctions are

- $\lambda = 0$ with $\phi_0(x) = 1$,
- $\lambda = n^2$ for positive integers n, with $\phi_{n,1}(x) = \sin(nx)$ and $\phi_{n,2}(x) = \cos(nx)$.

Problem 2.4.4. If $s = -\lambda > 0$ then the general solution is

$$\phi(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}.$$

The boundary conditions tells us that

$$c_1 + c_2 = 0,$$
$$c_1 e^{\sqrt{s}L} - c_2 e^{-\sqrt{s}L} = 0.$$

Note that $c_1 \neq 0$, else $\phi(x) = 0$. If we assume $c_1 \neq 0$, then the boundary conditions tells us that $2e^{\sqrt{s}L} = -1$, a contradiction.

Problem 2.4.6. (a) To get the equilibrium temperature distribution $\mathcal{U}(x)$, we need to solve

$$k\frac{d^2\mathcal{U}}{dx^2} = 0$$

with boundary conditions $\mathcal{U}(-L) = \mathcal{U}(L)$ and $\mathcal{U}_x(-L) = \mathcal{U}_x(L)$, and initial condition $u(x,0) = f(x)$. The general solution is

$$\mathcal{U}(x) = c_1 + c_2 x,$$

and upon substituting the boundary conditions one sees that $\mathcal{U}(x) = c$ for some constant c. We now use the conservation of heat energy, as discussed in Section 1.4 of the book, to get

$$\int_{-L}^{L} \mathcal{U}(x)\,dx = \int_{-L}^{L} u(x,0)\,dx,$$

giving us

$$\mathcal{U}(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

(b) Use equations (2.4.38) and (2.4.43) to get

$$\mathcal{U}(x) = \lim_{t\to\infty} u(x,t) = a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

Problem 2.4.7 (b). Separate variables to get the two ODEs: $G'(t) = -k\lambda G$ and $\phi'' = -\lambda\phi$. The time ODE has the same solution as always:

$$G(t) = e^{-k\lambda t}$$

Now we have to split up the x ODE into cases; for $\lambda > 0$, we have

$$\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

From $u(0) = 0$ we get that $C_1 = 0$

$$\phi(x) = C_2 \sin\sqrt{\lambda}x$$
$$\phi'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$
$$\phi'(L) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$
$$\cos\sqrt{\lambda}L = 0$$

This gives us the restriction on λ:

$$\lambda_n = \left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right)^2, \; n = 0, 1, 2, ...$$

$\lambda = 0$ gives us only a trivial solution. For $\lambda < 0$, we get

$$\phi(x) = C_1 \cosh \sqrt{\lambda} x + C_2 \sinh \sqrt{\lambda} x$$

From $u(0) = 0$ we get $C_1 = 0$, and the second boundary condition gives us

$$C_2 \sqrt{\lambda} \cosh \sqrt{\lambda} = 0$$

This has no solutions, so there are no negative eigenvalues.

So, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin \frac{(n+1/2)\pi x}{L} e^{-k((n+1/2)\pi/L)^2 t}$$

Now, we can show that $\sin \frac{(n+1/2)\pi x}{L}$ are orthogonal:

$$\int_0^L \sin \frac{(n+1/2)\pi x}{L} \sin \frac{(m+1/2)\pi x}{L}\, dx = \frac{1}{2} \int_0^L \left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m+1)\pi x}{L} \right) dx$$
$$= \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$$

So, we can use the same trick to get the Fourier coefficients:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(n+1/2)\pi x}{L}\, dx$$

Problem 2.5.1 (b). Separate variables as $u(x,y) = h(x)\phi(y)$. Then the resulting ODEs are

$$h''(x) = \lambda h \ ; \ \phi''(y) = -\lambda \phi$$

The ϕ equation, with $\phi(0) = \phi(H) = 0$ is something we've solved many times before:

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2$$

$$\phi_n = \sin \frac{n\pi y}{H}$$

Then, for $h(x)$ we get

$$h(x) = C_1 \cosh \frac{n\pi(x-L)}{H} + C_2 \sinh \frac{n\pi(x-L)}{H}$$

The requirement that $h'(L) = 0$ gives us that $C_2 = 0$, and so

$$h(y) = C_1 \cosh \frac{n\pi(x-L)}{H}$$

Our general solution is then

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi(x-L)}{H} \sin \frac{n\pi y}{H}$$

Then, plugging in the one inhomogeneous boundary condition, we get that

$$g(y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{H} \sinh \frac{-n\pi L}{H} \sin \frac{n\pi y}{L}$$

And so,

$$A_n = \frac{2}{H \sinh(-n\pi L/H)} \int_0^H g(y) \sin \frac{n\pi y}{L}\, dx$$

Problem 2.5.3. We separate variables, to get $\phi''(\theta) = -\lambda\phi$ and $r^2 G'' + rG' - n^2 G = 0$. The θ is something we've already seen; the eigenvalues are $\lambda = n^2$ and the eigenfunctions are $\cos n\theta$ and $\sin n\theta$, with a constant eigenfunction when $n = 0$. For the r equation, we get

$$G(r) = C_1 r^n + C_2 r^{-n}\ ,\ n > 0$$

$$G(r) = C_3 + C_4 \ln r\ ,\ n = 0$$

Here, the requirement that $G(r)$ be finite as $r \to \infty$ gives us that $C_1 = C_4 = 0$. So then, our general solution is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{-n} \cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n} \sin n\theta$$

(a)

$$u(a,\theta) = \ln 2 + 4\cos 3\theta$$

By matching coefficients, we get that $A_0 = \ln 2$, and that $A_3 \cdot a^{-3} = 4$ or that $A_3 = 4a^3$. So,

$$u(r,\theta) = \ln 2 + 4\frac{a^3}{r^3}\cos 3\theta$$

(b) In general, we can get the Fourier series coefficients for f (because $\cos n\theta$ and $\sin n\theta$ are mutually orthogonal on $[-\pi,\pi]$):

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n = \frac{2a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta$$

$$B_n = \frac{2a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

Problem 2.5.5 (a). We again have $\phi(\theta) = -\lambda\phi$, subject to $\phi'(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$. First, let's look fro positive eigenvalues:

$$\phi(\theta) = C_1 \cos\sqrt{\lambda}\theta + C_2 \sin\sqrt{\lambda}\theta$$

But the condition $\phi'(0) = 0$ mandates that $C_2 = 0$, so we only have

$$\phi\left(\frac{\pi}{2}\right) = \cos\sqrt{\lambda}\frac{\pi}{2} = 0$$

$$\lambda_n = (2n-1)^2\ ,\ n = 1,2,3,..$$

There are no zero or negative eigenvalues. Then, for the r equation we get $r^2 G'' + rG' - (2n-1)^2 G = 0$, for which we get

$$G(r) = C_1 r^{2n-1} + C_2 r^{1-2n}\ ,\ n > 0$$

This time, we require that $G(r)$ be finite as $r \to 0$, which gives $C_2 = 0$. So then, our general solutions is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos{(2n-1)\pi\theta}$$

Then, the formula for A_n is

$$A_n = \frac{4}{\pi a^{2n-1}}\int_0^{\frac{\pi}{2}} f(\theta)\cos{(2n-1)\pi\theta}\, dx$$

Math 241 (Spring 2019) Homework 4 Solutions

February 28, 2019

Problem 3.2.1(c)(d)(g). The sketches are attached after Problem 8.

Problem 3.2.2. The sketches are attached after Problem 8. One determines the Fourier coefficients using Equation (3.2.2) in the book. We record the answers here.

(b) One should get

$$\begin{aligned} a_0 &= \frac{1}{2L}(e^L - e^{-L}), \\ a_n &= \frac{(-1)^n L}{n^2\pi^2 + L^2}(e^L - e^{-L}), \\ b_n &= \frac{(-1)^n n\pi}{n^2\pi^2 + L^2}(e^L - e^{-L}). \end{aligned}$$

(c) One should get $b_1 = 1$ and all other coefficients zero.

(g) One should get

$$\begin{aligned} a_0 &= \frac{3}{2}, \\ a_n &= 0, \\ b_n &= \frac{1}{n\pi}(1 - \cos(n\pi)). \end{aligned}$$

Problem 3.3.1(c)(d). The sketches are attached after Problem 8.

Problem 3.3.5(b). The sketches are attached after Problem 8. One determines the Fourier coefficients using Equations (3.3.19) and (3.3.20). The answers are

$$a_0 = \frac{7}{6}, \qquad a_n = \frac{2}{n\pi}\left(3\sin\frac{n\pi}{2} - 2\sin\frac{n\pi}{6}\right).$$

Problem 3.3.18. (a) $f(x)$ will equal its Fourier series for all x, $-L \le x \le L$, if it is continuous so long as $f(-L) = f(L)$.

(b) $f(x)$ will equal its Fourier sine series for all x, $0 \le x \le L$, if $f(0) = f(L) = 0$.

(c) $f(x)$ will always equal its Fourier cosine series for $0 \le x \le L$ if it is continuous.

Problem 3.6.1. If $n \neq 0$:

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx \\ &= \frac{1}{2L\Delta}\int_{x_0}^{x_0+\Delta} e^{in\pi x/L}\,dx \\ &= \frac{1}{2L\Delta}\frac{L}{in\pi}\; e^{in\pi x/L}\Big|_{x_0}^{x_0+\Delta} \\ &= \frac{1}{2in\pi\Delta}\left(e^{in\pi(x_0+\Delta)/L} - e^{in\pi x_0/L}\right) \\ &= \frac{1}{2in\pi\Delta}e^{in\pi x_0/L}\left(e^{in\pi\Delta/L} - 1\right) \end{aligned}$$

If $n = 0$:

$$c_0 = \frac{1}{2L}\int_0^L f(x)\,dx = \frac{1}{2L\Delta}\int_{x_0}^{x_0+\Delta} dx$$

$$c_0 = \frac{1}{2L}$$

Problem 3.6.2. Let $f(x)$ be real on $[-L, L]$. Then,

$$c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

But we know that $\overline{f(x)} = f(x)$, because it's real, and that $\overline{e^{in\pi x/L}} = e^{-in\pi x/L}$. So,

$$c_{-n} = \frac{1}{2L}\int_{-L}^{L} \overline{f(x)e^{in\pi x/L}}\,dx$$

$$c_{-n} = \overline{\frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx} = \overline{c_n}$$

Problem 8. (b) The Fourier Cosine Series coefficients look like:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L x^2\,dx \\ &= \frac{1}{L}\frac{L^3}{3} \\ &= \frac{L^2}{3} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L x^2\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\left[\frac{L}{n\pi}x^2\sin\frac{n\pi x}{L}\bigg|_0^L - \frac{2L}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}\,dx\right] \\ &= -\frac{4}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}\,dx \\ &= -\frac{4}{n\pi}\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right] \end{aligned}$$

the second term is zero

$$\begin{aligned} &= \frac{4L}{n^2\pi^2}L\cos n\pi \\ &= \frac{4L^2(-1)^n}{n^2\pi^2} \end{aligned}$$

So, we have

$$g(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}$$

(c) Let's plug in at $x = L$:

$$g(L) = L^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2L^2}{3} = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Math 241 (Spring 2019) Homework 5 Solutions

March 14, 2019

Problem 4.2.1. (a) The ODE we want to solve is

$$T_0 \frac{\partial^2 u_E}{\partial x^2} - g\rho_0 = 0$$

with boundary conditions $u(0) = u(L) = 0$. By integration, the solution is

$$u_E(x) = \frac{g\rho_0}{2T_0}x^2 - \frac{g\rho_0 L}{2T_0}x.$$

(b) By assumption

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - g$$

and

$$c^2 \frac{\partial^2 u_E}{\partial x^2} = g, \qquad \frac{\partial^2 u_E}{\partial t^2} = 0.$$

Therefore

$$c^2 \frac{\partial^2 (u - u_E)}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + g - g = \frac{\partial^2 (u - u_E)}{\partial t^2},$$

as desired.

Problem 4.2.5. Use the same derivation in Section 4.2 of the book, assuming $\rho_0(x)$ and $T(x,t)$ are constants.

Problem 4.4.3(b). We show that the solution is

$$u(x,t) = e^{-\frac{\beta}{2\rho_0}t} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right),$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad B_n = \frac{2}{\omega_n L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

Let $u = \phi(x)h(t)$. By separating variables, one gets the ODEs

$$\frac{\rho_0}{T_0} \frac{d^2 h}{dt^2} + \frac{\beta}{T_0} \frac{dh}{dt} = -\lambda h$$

and

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi, \qquad \phi(0) = \phi(L) = 0.$$

The second ODE is something we have seen before; the eigenvalues and eigenfunctions are

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \;\; n = 1, 2, 3, \ldots$$
$$\phi_\lambda(x) = \sin \frac{n\pi x}{L}.$$

We now solve the first ODE. The characteristic polynomial, depending on λ, is

$$\frac{\rho_0}{T_0}\alpha^2 + \frac{\beta}{T_0}\alpha + \left(\frac{n\pi}{L}\right)^2 = 0,$$

implying

$$\alpha = -\frac{\beta}{2\rho_0} \pm \sqrt{\frac{\beta^2}{4\rho_0^2} - \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}.$$

By assumption on β the expression under the square root is negative, so the general solution for this ODE, depending on λ, is

$$h(t) = c_1 e^{-\frac{\beta}{2\rho_0}t}\sin(\omega_n t) + c_2 e^{-\frac{\beta}{2\rho_0}t}\cos(\omega_n t),$$

where

$$\omega_n = \sqrt{-\frac{\beta^2}{4\rho_0^2} + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}.$$

Putting everything together, the solution for the damped vibrating string is

$$u(x,t) = e^{-\frac{\beta}{2\rho_0}t}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right).$$

It remains to solve for A_n and B_n. Since $u(x,0) = f(x)$,

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

We use the usual orthogonality relations to get

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Finally, as $u_t(x,0) = g(x)$,

$$g(x) = \sum_{n=1}^{\infty} B_n\omega_n \sin\frac{n\pi x}{L},$$

so

$$B_n = \frac{2}{\omega_n L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Problem 4.4.7. As $g(x) = 0$, the solution of the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

(See Equations (4.4.11) and (4.4.13).) Let

$$F(y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{L}.$$

By taking $\alpha = n\pi x/L$ and $\beta = n\pi ct/L$ in the identity

$$\sin\alpha\cos\beta = \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2}$$

one immediately sees that

$$u(x,t) = \frac{F(x-ct) + F(x+ct)}{2}.$$

Problem 4.4.9.

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right] \\
&= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\,dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\,dx \\
&= c^2\int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\,dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\,dx \\
&= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx \\
&= c^2\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right) dx \\
&= c^2 \left.\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right|_0^L
\end{aligned}$$

Problem 4.4.10. (a) If $u(0) = u(L) = 0$, then this also means that $\frac{\partial u}{\partial t}(0) = \frac{\partial u}{\partial t}(L) = 0$. Hence, $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$, and so the energy is constant.

(b) In this case, $\frac{\partial u}{\partial t}(L) = 0$, and $\frac{\partial u}{\partial x}(0) = 0$, so both terms in $\frac{\mathrm{d}E}{\mathrm{d}t}$ are zero, so again energy is constant.

(c)

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= -\gamma c^2 u(L)\frac{\partial u}{\partial t}(L) \\
&= -\frac{\gamma c^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(u(L,t)^2\right)
\end{aligned}$$

So, integrating this gives that

$$E(t) = -\frac{\gamma c^2}{2}u(L,t)^2 + C$$

Hence, the total energy will decrease over time, if γ is positive.

(d) The total energy will increase over timeif γ is negative.

Problem 4.4.11. First, note that if $u(x,t) = R(x - ct)$, then we have that $\frac{\partial u}{\partial x} = R'(x - ct)$, and that $\frac{\partial u}{\partial t} = -cR'(x - ct)$ (just by the chain rule). The energies are then

$$\mathrm{KE} = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{c^2}{2}\int_0^L (R'(x-ct))^2\,dx$$

$$\mathrm{PE} = \frac{c^2}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{c^2}{2}\int_0^L (R'(x-ct))^2\,dx$$

Problem 8. The solution to this particular problem is done in the book in section 4.4, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right)$$

The formulas to find the coefficients are also given in the book:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

(a) Since f(x) = 0, all the A_n's will be zero. By matching coefficients, $B_5 \frac{5\pi c}{L} = 2$, so $B_5 = \frac{2L}{5\pi c}$, and all other B_n's are zero. Overall,

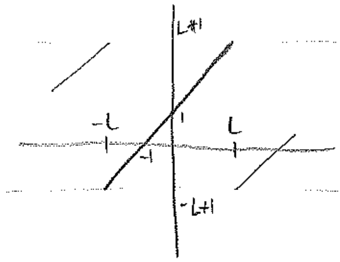
$$u(x,t) = \frac{2L}{5\pi c} \sin\frac{5\pi x}{L} \sin\frac{5\pi ct}{L}$$

(b) Again, by matching coefficients, $A_1 = 2$, $A_3 = 1$, and all other A_n's are zero. Also, $B_4 \frac{4\pi c}{L} = 3$, so $B_4 = \frac{3L}{4\pi c}$. So,

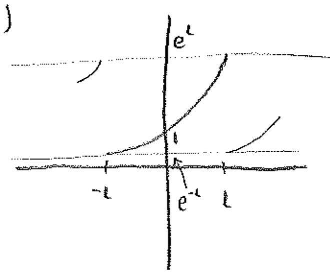
$$u(x,t) = 2\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + \frac{3L}{4\pi c}\sin\frac{4\pi x}{L}\sin\frac{4\pi ct}{L}$$

3.2.1

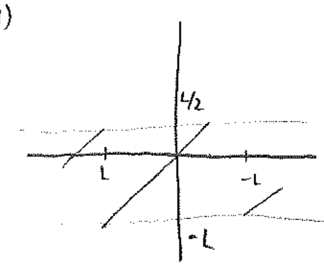
(c)



(d)



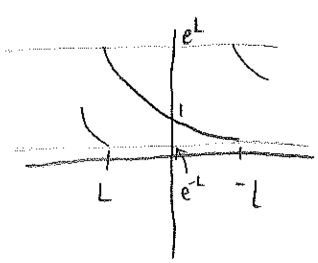
(g)



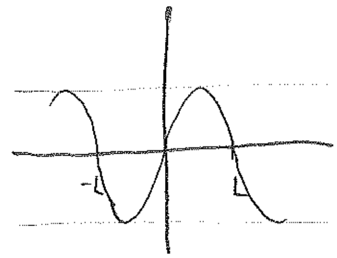
(extend periodically!)

3.2.2

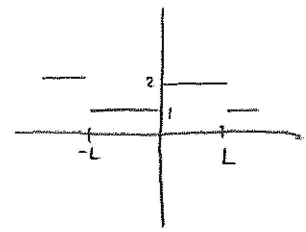
(b)



(c)



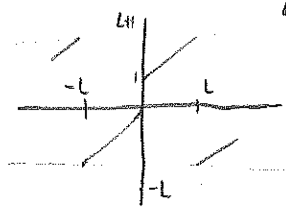
(g)



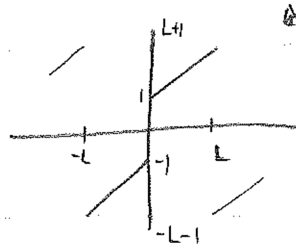
(extend periodically!)

3.3.1

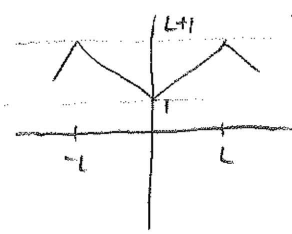
(c)



← Usual Fourier

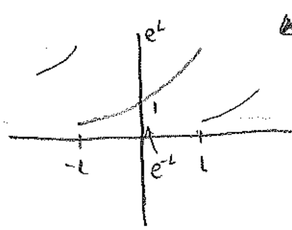


← Fourier Sin

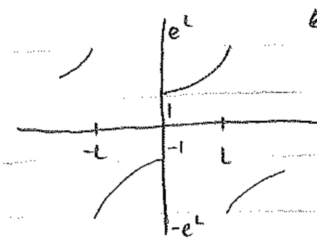


← Fourier Cos

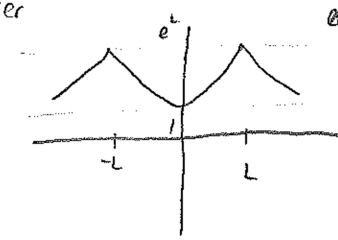
(d)



← Usual Fourier



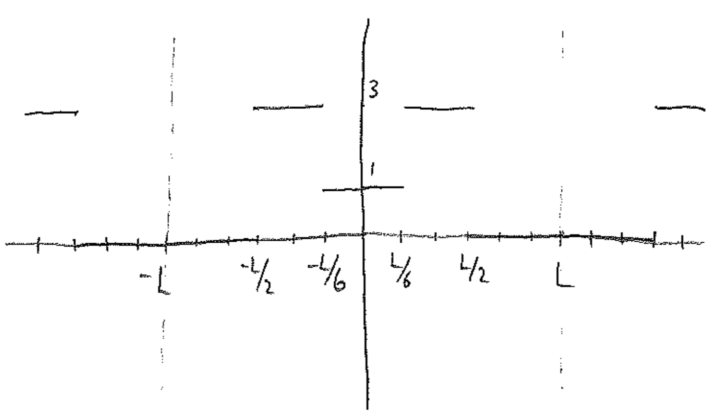
← Fourier Sin



← Fourier Cos

3.3.5

(b)



Math 241 (Spring 2019) Homework 6 Solutions

March 21, 2019

Problem 5.3.2. (b) Letting $u(x,t) = \phi(x)h(t)$, one gets

$$\frac{h_{tt}}{h} - \frac{\beta h_t}{\rho h} = T_0 \frac{\phi_{xx}}{\rho\phi} + \frac{\alpha}{\rho}.$$

Thus in order to separate variables we need $\beta\rho = c$ for some constant c.

(c) If $\beta = c\rho$ as in the previous part, then

$$h_{tt} - ch_t = -\lambda h$$
$$T_0\phi_{xx} + \alpha\phi = -\lambda\rho\phi.$$

The spatial equation is in Sturm-Liouville form (with $p(x) = T_0$, $q(x) = \alpha(x)$, and $\sigma(x) = \rho(x)$). The time equation can be solved by considering the characteristic polynomial

$$y^2 - cy + \lambda = 0$$

with zeros

$$y = \frac{c \pm \sqrt{c^2 - 4\lambda^2}}{2}.$$

If $c^2 - 4\lambda^2 > 0$ then the general solution is

$$h(t) = c_1 e^{(\frac{c+\sqrt{c^2-4\lambda^2}}{2})t} + c_2 e^{(\frac{c-\sqrt{c^2-4\lambda^2}}{2})t}.$$

If $c^2 - 4\lambda^2 = 0$ then the general solution is

$$h(t) = c_1 e^{ct/2} + c_2 t e^{ct/2}.$$

If $c^2 - 4\lambda^2 < 0$ then the general solution is

$$h(t) = c_1 e^{ct/2} \cos\Big(\sqrt{4\lambda^2 - c^2}t\Big) + c_2 e^{ct/2} \sin\Big(\sqrt{4\lambda^2 - c^2}t\Big).$$

Problem 5.3.3. By comparing coefficients with the general Sturm-Liouville equation, one needs

$$p = H, \qquad p' = \alpha H, \qquad q = \gamma H, \qquad \sigma = \beta H.$$

The first two equations tells us that $H' = \alpha H$, so by the integration factor

$$H(x) = ce^{\int \alpha(x)\,dx}.$$

We can let $c = 1$, and pick

$$H(x) = e^{\int \alpha(x)\,dx}, \qquad p = H, \qquad q = \gamma H, \qquad \sigma = \beta H.$$

Problem 5.3.5. From Table 2.4. the eigenvalues and corresponding eigenfunctions are

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, \cdots,$$
$$\phi(x) = \cos\frac{n\pi x}{L}.$$

Thus (a) is clear. For (b), one observes that $\phi(x)$ has zeros

$$\frac{L}{2n}, \frac{3L}{2n}, \cdots, , \frac{(1+2(n-1))L}{2n}$$

on the interval $(0, L)$, so the $(n+1)^{th}$-eigenfunctions has n zeros. (Warning: For this problem $\lambda_1 = 0$, $\lambda_2 = (\pi/L)^2$, $\lambda_3 = (2\pi/L)^2$, and in general $\lambda_{n+1} = (n\pi/L)^2$ has one less index!)

For part (c), any piecewise smooth function on $(0, L)$ can be approximated by these eigenfunctions by considering the Fourier cosine series, and the eigenfunctions are orthogonal by the orthogonality relations from Chapter 3. For (d), the Rayleigh quoient in this case is

$$\lambda = \frac{\int_0^L (d\phi/dx)^2\, dx}{\int_0^L \phi^2\, dx},$$

which tells us that the eigenvalues must be nonnegative. It also tells us that $\lambda = 0$ is possible, since this implies the numerator of the Rayleigh quotient is zero, and thus $(d\phi/dx)^2 = 0$, telling us that $\phi(x)$ can be chosen to be a nonzero constant.

Problem 5.3.6. Proceed exactly the same as in the previous problem. The only difference is that the eigenvalues in this problem are "shifted":

$$\lambda = \left(\frac{n\pi - \pi/2}{L}\right)^2, \qquad n = 1, 2, 3 \cdots.$$

Problem 5.3.9. (a) Multiplying by $\frac{1}{x}$ gives the equation

$$x\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}\phi}{\mathrm{d}x} + \frac{\lambda}{x}\phi = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + \frac{\lambda}{x}\phi = 0$$

Which is in regular Sturm-Liouville form, with $p(x) = x$, $q(x) = 0$, and $\sigma(x) = \frac{1}{x}$.

(b) Using the Rayleigh quotient,

$$\lambda = \frac{-x\phi\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_1^b + \int_1^b x\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 dx}{\int_1^b \phi^2(1/x)\, dx}$$

the first term vanishes because of the boundary conditions:

$$\lambda = \frac{\int_1^b x\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 dx}{\int_1^b \phi^2(1/x)\, dx}$$

Since the numerator and denominator are integrals of a non-negative quantity, $\lambda \geq 0$.

(c) Guess a solution of the form $\phi(x) = x^r$. This gives $r^2 = -\lambda$; since $\lambda \geq 0$, this means r is imaginary; the solution is therefore

$$\phi(x) = C_1 \cos(\sqrt{\lambda}\ln x) + C_2 \sin(\sqrt{\lambda}\ln x)$$

Plugging in boundary conditions, $\phi(1) = 0$ gives $C_1 = 0$, while $\phi(b) = 0$ tells us the eigenvalues:

$$\sin(\sqrt{\lambda}\ln b) = 0$$

$$\sqrt{\lambda}\ln b = n\pi$$
$$\lambda = \left(\frac{n\pi}{\ln b}\right)^2$$

We have to look at $\lambda = 0$ separately: then we get $r^2 = 0$; the solution therefore $\phi(x) = A + B\ln x$. But $\phi(1) = A = 0$, and $\phi(b) = B\ln b = 0$ has no non-trivial solution, so 0 is not an eigenvalue.

(d) The eigenfunctions are orthogonal with respect to $\frac{1}{x}$ as a weight.

$$\int_1^b \phi_n\phi_m\sigma\,dx = \int_1^b \sin\frac{n\pi\ln x}{\ln b}\sin\frac{m\pi\ln x}{\ln b}\frac{1}{x}\,dx$$

Let $u = \ln x$, $du = \frac{1}{x}dx$, and the bounds become $u : 0 \to \ln b$.

$$\int_1^b \phi_n\phi_m\sigma\,dx = \int_0^{\ln b} \sin\frac{n\pi u}{\ln b}\sin\frac{m\pi u}{\ln b}\,du$$

Which we know is 0 unless $m = n$, as they are the familiar sine eigenfunctions, with $L = \ln b$.

(e) The nth eigenfunction is $\sin\frac{n\pi\ln x}{\ln b}$. It has zeros when:

$$\frac{n\pi\ln x}{\ln b} = k\pi$$
$$\ln x = \ln b\left(\frac{k}{n}\right)$$
$$x = e^{\frac{k}{n}\ln b} = b^{\frac{k}{n}}$$

x is within the interval $[1, b]$ when $0 < k < n$; there are $n - 1$ such values, and hence ϕ_n has $n - 1$ zeros.

Problem 5.4.2(b). The time ODE is simple, $h(t) = e^{-\lambda t}$. The x ODE looks like

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[K_0(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}\right] + \lambda c\rho\phi = 0$$

which is Sturm-Liouville form with $p = 1$, $q = 0$, and $\sigma = c\rho$. Using completeness the general solution is a linear combination of the separated solutions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n t}$$

And orthogonality means that

$$a_n = \frac{\int_0^L f(x)\phi_n(x)c\rho\,dx}{\int_0^L \phi_n^2(x)c\rho\,dx}$$

.

As for $\lim_{t\to\infty} u(x,t)$, this will only be non-zero if a constant solution is allowed. For these boundary conditions, it is allowed, so zero is an eigenvalue, and $\phi_1(x) = 1$. Hence $\lim_{t\to\infty} u(x,t)$ will be a (possibly non-zero) constant.

Problem 5.4.3. Separate variables: $u(r,t) = \phi(r)h(t)$. Then,

$$\phi h' = \frac{k}{r}\frac{\partial}{\partial r}(r\phi' h)$$
$$\frac{h'}{kh} = \frac{1}{r\phi}\frac{\partial}{\partial r}(r\phi') = -\lambda$$

The time equation gives that $h(t) = e^{-k\lambda t}$. The r equation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \lambda r\phi = 0$$

Which is a Sturm-Liouville problem with $p = r$, $q = 0$, and $\sigma = r$. So, we get that

$$u(r,t) = \sum_{n=1}^{\infty} a_n \phi_n(r) e^{-k\lambda t}$$

$$a_n = \frac{\int_0^a f(r)\phi_n(r) r \, dr}{\int_0^a \phi_n^2(r) r \, dr}$$

Problem 8. For this problem, $p(x) = 1 + x^2$, $q(x) = 0$, and $\sigma(x) = 1$. So, the Rayleigh quotient is

$$\lambda = \frac{-(1+x^2)\phi \frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_0^L + \int_0^L (1+x^2) \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 dx}{\int_0^L \phi^2 \, dx}$$

The first term is 0 because of the boundary conditions, so

$$\lambda = \frac{\int_0^L (1+x^2) \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 dx}{\int_0^L \phi^2 \, dx}$$

The numerator and denominator are both integrals of non-negative quantities, so λ must be non-negative. $\lambda = 0$ when the integrand of the numerator is zero; since $1 + x^2 > 0$ this must mean $\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0$. In turn this would mean $\phi = C$, a constant. But the boundary conditions force $C = 0$, which makes ϕ trivial. Therefore, zero is not an eigenvalue, and $\lambda > 0$.

Math 241 (Spring 2019) Homework 7 Solutions

March 28, 2019

Problem 7.3.1(a). By the exact same derivation as Section 7.3 of the text,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L} e^{-\lambda_{nm}kt},$$

where

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

Since the eigenvalues are positive,

$$\lim_{t\to\infty} u(x,y,t) = 0.$$

Problem 7.3.2(b). By separation the variables successively as in the text:

$$\begin{aligned} u(x,y,z,t) &= \phi(x,y,z)h(t) \\ \phi(x,y) &= e(x)\alpha(y,z) \\ \alpha(y,z) &= f(y)g(z) \end{aligned}$$

one gets the following ODEs up to reordering:

$$\begin{aligned} h'(t) &= -\lambda k h(t), \\ e''(x) &= -\mu e(x), \qquad e'(0) = e'(L) = 0, \\ f''(y) &= -\sigma f(y), \qquad f'(0) = f'(H) = 0 \\ g''(z) &= -(\lambda - \mu - \sigma) g(z), \qquad g'(0) = g'(W) = 0. \end{aligned}$$

The general solution for $h(t)$ is $h(t) = e^{-\lambda kt}$. Using Table 2.4.1, one sees that the eigenvalues and eigenfunctions are

$$\lambda_{m,n,l} = \left(\frac{n\pi}{L}\right)^2, + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{l\pi}{W}\right)^2 \quad (n,m,l \geq 0), \qquad \phi_{m,n,l}(x,y,z) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi z}{W}\right).$$

Therefore

$$u(x,y,z,t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mnl} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi z}{W}\right) e^{-\lambda kt}.$$

We compute the coefficients for A_{nml} using the standard equation

$$A_{nml} = \frac{\int_0^W \int_0^H \int_0^L \alpha(x,y,z)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{l\pi z}{W}\right)\, dzdydx}{\int_0^W \int_0^H \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{H}\right)\cos^2\left(\frac{l\pi z}{W}\right)\, dzdydx}.$$

The denominator can be further simplified using the orthogonality relation

$$\int_0^T \cos^2\frac{p\pi u}{T}\, du = \begin{cases} T, & p = 0 \\ T/2 & p > 0. \end{cases}$$

In particular, since $\lambda_{000} = 0$ and λ is positive otherwise,

$$\lim_{t\to\infty} u(x,y,z,t) = A_{000} = \frac{1}{LHW}\int_0^W \int_0^H \int_0^L \alpha(x,y,z)\, dzdydx.$$

Problem 7.3.4(b). In this problem one has

$$h''(t) = -\lambda k h(t),$$
$$\nabla^2 \phi = -\lambda \phi.$$

By the same derivation as Section 7.3 of the text, separate $\phi(x,y) = f(x)g(y)$ to get the eigenvalues and eigenfunctions

$$\lambda_{m,n} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \ \ (n,m \geq 0), \qquad \phi_{m,n,l}(x,y,z) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right).$$

For $\lambda_{00} = 0$ one has $h(t) = c_1 + c_2 t$, with $c_1 = h(0) = 0$, and for the other eigenvalues, which are positive, $h(t) = c_1 \sin\left(c\sqrt{\lambda}t\right) + \cos\left(c\sqrt{\lambda}t\right)$ with $c_2 = h(0) = 0$. Thus

$$u(x,y,t) = A_{00}t + \sum_{(n,m)\neq(0,0)} A_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin\left(c\sqrt{\lambda_{n,m}}t\right).$$

Since

$$\alpha(x,y) = u_t(x,y,0) = A_{0,0} + \sum_{(n,m)\neq(0,0)} cA_{nm}\sqrt{\lambda_{n,m}}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right),$$

the orthogonality relations for the cosine function tells us that

$$A_{0,0} = \frac{1}{HL}\int_0^L\int_0^H \alpha(x,y)\,dydx$$
$$A_{0,m} = \frac{2}{cHL\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\cos\left(\frac{m\pi y}{H}\right)\,dydx \ \ (m \geq 1)$$
$$A_{n,0} = \frac{2}{cHL\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\cos\left(\frac{n\pi x}{L}\right)\,dydx \ \ (n \geq 1)$$
$$A_{n,m} = \frac{4}{cHL\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\,dydx \ \ (n,m \geq 1).$$

Problem 7.3.5(b). By separating variables, one gets

$$h''(t) + kh'(t) + \lambda c^2 h(t) = 0$$
$$f''(x) + \mu f(x) = 0$$
$$g''(y) + (\lambda - \mu)g(y) = 0.$$

Problem 7.4.1. (a) We can separate variables in this PDE: if we let $\phi(x,y) = f(x)g(y)$, then we get

$$f''g + fg'' + \lambda fg = 0$$

$$\frac{f''}{f} + \lambda = -\frac{g''}{g} = \mu$$

In y, our ODE reads $g''(y) = -\mu g(y)$, with the boundary conditions $g(0) = g(H) = 0$, so

$$g_n(y) = \sin\frac{n\pi y}{H}$$

$$\mu_n = \left(\frac{n\pi}{H}\right)^2, \ n = 1,2,3,...$$

Then plugging into our ODE for f gives $f''(x) = -(\lambda - \mu_n)f(x)$; $f'(0) = f'(L) = 0$. The solutions is then

$$\lambda_{mn} - \mu_n = \left(\frac{m\pi}{L}\right)^2, \ m = 0,1,2,...$$

$$f_m(x) = \begin{cases} \cos\frac{m\pi x}{L}, & m > 0 \\ 1, & m = 0 \end{cases}$$

So, the doubly infinite set of eigenvalues is

$$\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, \; m = 0, 1, 2, ...; \; n = 1, 2, 3, ...$$

(b) When $L = H$, $\lambda_{mn} = \frac{(n^2+m^2)\pi^2}{L^2}$, with m starting at zero and n starting at one. For any m and n, except the ones where $m = 0$, $\lambda_{mn} = \lambda_{nm}$, so most of the eigenvalues have at least two associated eigenfunctions.

(c) The eigenfunctions are $\phi_{mn}(x, y) = \cos\frac{m\pi x}{L}\sin\frac{n\pi y}{H}$.

$$\begin{aligned}\int_0^L\int_0^H \phi_{mn}\phi_{kl}\,dxdy &= \int_0^L\int_0^H \cos\frac{m\pi x}{L}\sin\frac{n\pi y}{H}\cos\frac{k\pi x}{L}\sin\frac{l\pi y}{H}\,dxdy \\ &= \int_0^L \cos\frac{m\pi x}{L}\cos\frac{k\pi x}{L}\,dx\int_0^H \sin\frac{n\pi y}{H}\sin\frac{l\pi y}{H}\,dy \\ &= \begin{cases} \frac{LH}{4}, & m = k \text{ and } n = l \\ 0, & o.w. \end{cases}\end{aligned}$$

Problem 7.4.2. Using the Rayleigh quotient,

$$\lambda = \frac{-\oint \phi\nabla\phi\cdot\hat{\mathbf{n}}\,ds + \iint_R |\nabla\phi|^2\,dx\,dy}{\iint_R \phi^2\,dx\,dy}$$

Our boundary conditions say that ϕ is zero everywhere on the boundary, so the first integral over the boundary will vanish. Then we are left with

$$\lambda = \frac{\iint_R |\nabla\phi|^2\,dx\,dy}{\iint_R \phi^2\,dx\,dy}$$

Since the numerator and denominator are integrals of non-negative quantities (since they are squared), λ is also non-negative.

Problem 7. (a) Plug into the PDE, which tells us that $\nabla^2\phi_{nm} = -\lambda_{nm}\phi_{nm}$.

$$\nabla^2\phi_{nm} = \frac{\partial^2\phi_{nm}}{\partial x^2} + \frac{\partial^2\phi_{nm}}{\partial y^2}$$

$$\nabla^2\phi_{nm} = -n^2\pi^2\sin(n\pi x)\sin(m\pi y) + m^2\pi^2\sin(m\pi x)\sin(n\pi y) - m^2\pi^2\sin(n\pi x)\sin(m\pi y) + n^2\pi^2\sin(m\pi x)\sin(n\pi y)$$

$$\nabla^2\phi_{nm} = -\pi^2(n^2+m^2)\left((\sin(n\pi x)\sin(m\pi y) - \sin(m\pi x)\sin(n\pi y)\right)$$

So, $\lambda_{mn} = \pi^2(n^2 + m^2)$.

(b) We know that for the wave equation, the time dependent part looks like $h(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t$. So, the general solution is

$$u(x, y, t) = \sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\Big((\sin(n\pi x)\sin(m\pi y) - \sin(m\pi x)\sin(n\pi y)\Big)\Big(A_{nm}\cos\pi\sqrt{n^2+m^2}t + B_{nm}\sin\pi\sqrt{n^2+m^2}t\Big)$$

(c) At $t = 0$, we get that

$$u(x, y, 0) = f(x, y) = \sum_{m=2}^{\infty}\sum_{n=1}^{m-1} A_{nm}\phi_{nm}(x, y)$$

Using formula (7.4.14), we get that

$$A_{nm} = \frac{\iint_R f\phi_{nm}\,dx\,dy}{\iint_R \phi_{nm}^2\,dx\,dy}$$

$$A_{nm} = \frac{\int_0^1 \int_0^x f\phi_{nm}\,dy\,dx}{\int_0^1 \int_0^x \phi^2\,dy\,dx}$$

$$A_{nm} = 4\int_0^1 \int_0^x f(x,y)\phi_{nm}(x,y)\,dy\,dx$$

Taking the time derivative at time 0, we get that

$$u_t(x,y,0) = g(x,y) = \sum_{m=2}^{\infty}\sum_{n=1}^{m-1} B_{nm}\pi\sqrt{n^2+m^2}\phi_{mn}(x,y)$$

$$B_{nm} = \frac{4}{\pi\sqrt{n^2+m^2}}\int_0^1 \int_0^x g(x,y)\phi_{nm}(x,y)\,dy\,dx$$

Problem 8.

$$E(t) = \frac{1}{2}\iiint_\Omega (u_t^2 + u_x^2 + u_y^2 + u_z^2 + u^2)\,dV$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\iiint_\Omega (2u_tu_{tt} + 2u_xu_{xt} + 2u_yu_{yt} + 2u_zu_{zt} + 2uu_t)\,dV$$

From the PDE, we get

$$\begin{aligned}
&= \iiint_\Omega (u_tu_{xx} + u_tu_{yy} + u_tu_{zz} - uu_t + u_xu_{xt} + u_yu_{yt} + u_zu_{zt} + uu_t)\,dV \\
&= \iiint_\Omega (u_tu_{xx} + u_xu_{xt} + u_tu_{yy} + u_yu_{yt} + u_tu_{zz} + u_zu_{zt})\,dV \\
&= \iiint_\Omega \big((u_tu_x)_x + (u_tu_y)_y + (u_tu_z)_z\big)\,dV \\
&= \iiint_\Omega \nabla\cdot\langle u_tu_x, u_tu_y, u_tu_z\rangle\,dV
\end{aligned}$$

Using the divergence theorem, this is

$$\begin{aligned}
&= \iint_{bdry\ \Omega} \langle u_tu_x, u_tu_y, u_tu_z\rangle\cdot\hat{\mathbf{n}}\,dA \\
&= \iint_{bdry\ \Omega} u_t\nabla u\cdot\hat{\mathbf{n}}\,dA
\end{aligned}$$

From our boundary conditions, we know that $\nabla u\cdot\hat{\mathbf{n}} = 0$ everywhere on the boundary, so this integral is zero; hence E is a constant.

Math 241 (Spring 2019) Homework 8 Solutions

April 11, 2019

Problem 7.7.1. By separation of variables and solving ODEs, exactly like in Section 7.1, the general solution is given by

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}r)\left(C_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)+D_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)\right)(A_{mn}\sin m\theta+B_{mn}\cos m\theta).$$

We now solve for the coefficients. Since $u(r,\theta,0)=0$, one can eliminate the $\cos\left(c\sqrt{\lambda_{mn}}t\right)$ terms, and

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}r)\sin\left(c\sqrt{\lambda_{mn}}t\right)(A_{mn}\sin m\theta+B_{mn}\cos m\theta).$$

The initial condition tells us that

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}r)\cdot c\sqrt{\lambda_{mn}}\cdot(A_{mn}\sin m\theta+B_{mn}\cos m\theta)=\alpha(r)\sin 3\theta,$$

with

$$\lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2.$$

Thus, by the orthogonality relation with respect to θ (or by observation), one has $B_{mn}=0$ for all m, and $A_{mn}=0$ if $m\neq 3$. When $m=3$ the orthogonality relation for the Bessel function tells us that

$$A_{3n}=\frac{\int_0^a\alpha(r)J_3(\sqrt{\lambda_{3n}}r)r\,dr}{c\sqrt{\lambda_{3n}}\int_0^a J_3^2(\sqrt{\lambda_{3n}}r)r\,dr},$$

and

$$u(r,\theta,t)=\sum_{n=1}^{\infty}A_{3n}J_3(\sqrt{\lambda_{3n}}r)\sin\left(c\sqrt{\lambda_{3n}}t\right)\sin 3\theta.$$

Problem 7.7.2(a)(c). By separation of variables and solving ODEs, exactly like in Section 7.1, the general solution is given by

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\sqrt{\lambda_{mn}}r)\left(C_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)+D_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)\right)(A_{mn}\sin m\theta+B_{mn}\cos m\theta).$$

In this problem the λ_{mn} are slightly different however: the condition $u_r(a,\theta,t)=0$ tells us that

$$\left(\frac{y_{mn}}{a}\right)^2.$$

where y_{mn} are the zeros to $J_m'(\sqrt{\lambda}r)=0$.

(a) This part is almost the same as Problem 7.7.1, except sin is replaced by cos in the initial condition. The answer is

$$u(r,\theta,t)=\sum_{n=1}^{\infty}A_{5n}J_5(\sqrt{\lambda_{5n}}r)\sin\left(c\sqrt{\lambda_{5n}}t\right)\cos 5\theta$$

where

$$A_{5n} = \frac{\int_0^a \alpha(r) J_5(\sqrt{\lambda_{5n}}r) r\, dr}{c\sqrt{\lambda_{mn}} \int_0^a J_5^2(\sqrt{\lambda_{5n}}r) r\, dr}.$$

(c) In this case $u_t(r,\theta,0) = 0$ tells us that the terms with $\sin\left(c\sqrt{\lambda_{mn}}t\right)$ vanishes, and

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r) \cos\left(c\sqrt{\lambda_{mn}}t\right) (A_{mn}\sin m\theta + B_{mn}\cos m\theta).$$

The initial condition tells us that

$$\alpha(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r) (A_{mn}\sin m\theta + B_{mn}\cos m\theta)$$

and we apply orthogonality to see that

$$A_{mn} = \frac{\int_0^{2\pi}\int_0^a \alpha(r,\theta) J_m(\sqrt{\lambda_{mn}r}) r \sin m\theta\, dr d\theta}{\int_0^{2\pi}\int_0^a J_m^2(\sqrt{\lambda_{mn}r}) r \sin^2 m\theta\, dr d\theta},$$

$$B_{mn} = \frac{\int_0^{2\pi}\int_0^a \alpha(r,\theta) J_m(\sqrt{\lambda_{mn}r}) r \cos m\theta\, dr d\theta}{\int_0^{2\pi}\int_0^a J_m^2(\sqrt{\lambda_{mn}r}) r \cos^2 m\theta\, dr d\theta}.$$

Problem 7.7.6(c). The solution to this problem is exactly the same as Problem 7.7.1, except with slightly different eigenvalues λ_{mn}. We can copy the work done in Problem 7.7.1 if we show that there are infinitely many solutions λ of order one to

$$J_m(\sqrt{\lambda}a) - \sqrt{\lambda} J_m'(\sqrt{\lambda}a) = 0,$$

and the solutions are all positive. A way to do this is to apply the mean value theorem with respect to the zeros z_{mn} of J_m. (In fact this argument tells us that the solutions intertwines with the zeros of $J_m(\sqrt{\lambda}a)$ and $J_m'(\sqrt{\lambda}a)$.)

In summary the answer to this problem is

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_{3n} J_3(\sqrt{\lambda_{3n}}r) \sin\left(c\sqrt{\lambda_{3n}}t\right) \sin 3\theta$$

with

$$A_{mn} = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}}r) r\, dr}{c\sqrt{\lambda_{mn}} \int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r\, dr}, \qquad \lambda_{mn} = \left(\frac{\alpha_{mn}}{a}\right)^2,$$

where α_{mn} are the solutions to the equation $J_m(\sqrt{\lambda}a) - \sqrt{\lambda}J_m'(\sqrt{\lambda}a) = 0$ (of which there are infinitely many and all positive).

Problem 7.7.7. The setup of this problem is very similar to Problem 7.7.1. The main difference is that you get a first-order ODE for h after separating variables:

$$h'(t) = -\lambda k h(t).$$

After doing the same thing as before, the solution to this PDE is

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r) e^{-\lambda_{mn}kt} (A_{mn}\sin m\theta + B_{mn}\cos m\theta)$$

where

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

and z_{mn} are the zeros of J_m. The initial condition tells us that

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r) (A_{mn}\sin m\theta + B_{mn}\cos m\theta)$$

and we apply orthogonality to see that

$$A_{mn} = \frac{\int_0^{2\pi}\int_0^a f(r,\theta)J_m(\sqrt{\lambda_{mn}}r)r\sin m\theta\,drd\theta}{\int_0^{2\pi}\int_0^a J_m^2(\sqrt{\lambda_{mn}}r)r\sin^2 m\theta\,drd\theta},$$
$$B_{mn} = \frac{\int_0^{2\pi}\int_0^a f(r,\theta)J_m(\sqrt{\lambda_{mn}}r)r\cos m\theta\,drd\theta}{\int_0^{2\pi}\int_0^a J_m^2(\sqrt{\lambda_{mn}}r)r\cos^2 m\theta\,drd\theta}.$$

Problem 7.7.10. Separate variables $u(r,t) = f(r)h(t)$.

$$fh' = \frac{k}{r}\frac{\partial}{\partial r}(rf'h)$$

$$\frac{h'}{kh} = \frac{1}{rf}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = -\lambda$$

The time equation is simple to solve and gives $h(t) = e^{-k\lambda t}$. The r equation is

$$r\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{\mathrm{d}f}{\mathrm{d}r} + \lambda r f = 0$$

Multiplying by r gives

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + \lambda r^2 f = 0$$

Making the transformation $z = \sqrt{\lambda}r$ gives us Bessel's equation of order 0. So,

$$f(r) = C_1 J_0(\sqrt{\lambda}r) + C_2 Y_0(\sqrt{\lambda}r)$$

But Y_0 is singular at $r = 0$, so only the $J_0(\sqrt{\lambda}r)$ term can survive. The boundary condition $f(a) = 0$ will fix the eigenvalues:

$$J_0(\sqrt{\lambda}a) = 0$$

If z_n is the nth zero of J_0, then $\lambda_n = \left(\frac{z_n}{a}\right)^2$. Then, our general solution is

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}r)e^{-k\lambda_n t}$$

Plug in the initial conditions to get

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}r)$$

And so

$$a_n = \frac{\int_0^a f(r)J_0(\sqrt{\lambda_n}r)r\,dr}{\int_0^a J_0^2(\sqrt{\lambda_n}r)r\,dr}$$

Since zero is not an eigenvalue (because $J_0(0) \neq 0$), all terms in the series will decay as $t \to \infty$, and so $\lim_{t\to\infty} u(x,t) = 0$.

Problem 7.9.1(b). Separate variables; the θ equation as usual gives us $\mu = m^2$ and eigenfunctions $\cos m\theta$ and $\sin m\theta$. The r-equation looks like

$$r^2\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\frac{\mathrm{d}f}{\mathrm{d}r} + (\lambda r^2 - m^2)f = 0$$

The solutions to this are Bessel functions of order m, but the requirement that $|f(0)| < \infty$ eliminates the Y_m solutions. So, $f(r) = c_1 J_m(\sqrt{\lambda} r)$, and the other boundary condition fixes our eigenvalues:

$$J_m(\sqrt{\lambda} a) = 0$$

which gives us a set of eigenvalues λ_{mn}. Then, the z-equation is $\frac{\mathrm{d}^2 h}{\mathrm{d}z^2} = \lambda h$, with $h(H) = 0$. A good choice of eigenfunctions is $\sinh\left(\sqrt{\lambda_{mn}}(z - H)\right)$ and $\cosh\left(\sqrt{\lambda_{mn}}(z - H)\right)$, as our boundary condition eliminates the hyperbolic cosine term. So the general solution now looks like

$$u(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}} r) \sinh\left(\sqrt{\lambda_{mn}}(z - H)\right) (A_{mn} \cos m\theta + B_{mn} \sin m\theta)$$

Plug in $z = 0$ to get

$$u(r, \theta, 0) = \alpha(r) \sin 7\theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}} r) \sinh\left(-\sqrt{\lambda_{mn}} H\right) (A_{mn} \cos m\theta + B_{mn} \sin m\theta)$$

Matching coefficients, we get that all A_{mn}'s are zero, and that B_{mn} is non-zero only for $m = 7$. Then,

$$\alpha(r) = \sum_{n=1}^{\infty} B_{7n} \sinh\left(-\sqrt{\lambda_{7n}} H\right) J_7(\sqrt{\lambda_{7n}} r)$$

So,

$$B_7 n = \frac{1}{\sinh\left(-\sqrt{\lambda_{7n}} H\right)} \frac{\int_0^a \alpha(r) J_7(\sqrt{\lambda_{7n}} r) r \, dr}{\int_0^a J_7^2(\sqrt{\lambda_{7n}} r) r \, dr}$$

$$u(r, \theta, z) = \sum_{n=1}^{\infty} B_{7n} J_7(\sqrt{\lambda_{7n}} r) \sinh\left(\sqrt{\lambda_{7n}}(z - H)\right) \sin 7\theta$$

Problem 7.9.2(c). The θ-equation gives us eigenvalues $\mu = m^2$, with m starting at 0, and eigenfunctions $\cos m\theta$. The z-equation can also be solved easily; it looks like $\frac{\mathrm{d}^2 h}{\mathrm{d}z^2} = \lambda h$, with boundary conditions $h'(0) = h'(H) = 0$. So

$$\lambda_n = -\left(\frac{n\pi}{H}\right)^2$$

$$h_n(z) = \cos \frac{n\pi z}{H}$$

The r-equation looks like

$$r^2 \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r \frac{\mathrm{d}f}{\mathrm{d}r} + \left(-\left(\frac{n\pi}{H}\right)^2 r^2 - m^2\right) f = 0$$

Or, if we make the transformation $w = \frac{n\pi r}{H}$, it becomes

$$w^2 \frac{\mathrm{d}^2 f}{\mathrm{d}w^2} + w \frac{\mathrm{d}f}{\mathrm{d}w} + (-w^2 - m^2) f = 0$$

The solutions are K_m and I_m, but K_m is singular at the origin, so only I_m sticks around, and we get

$$f(r) = I_m\left(\frac{n\pi r}{H}\right)$$

Except when $n = 0$, in which case the equation is equidimensional and we get $f(r) = r^m$. And if $n = m = 0$, then we get just a constant. The general solution is then

$$u(r, \theta, z) = A_{00} + \sum_{m=1}^{\infty} A_{m0} r^m \cos m\theta + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} I_m\left(\frac{n\pi r}{H}\right) \cos \frac{n\pi z}{H} \cos m\theta$$

Plugging in the last boundary condition, we get that

$$\frac{\partial u}{\partial r} = \beta(\theta, z) = \sum_{m=1}^{\infty} m A_{m0} a^{m-1} \cos m\theta + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{n\pi}{H} A_{mn} I_m' \left(\frac{n\pi a}{H}\right) \cos \frac{n\pi z}{H} \cos m\theta$$

For m and n both non-zero, we get

$$A_{mn} = \frac{4}{n\pi^2 I_m'(n\pi a/H)} \int_0^{\pi} \int_0^{H} \beta(\theta, z) \cos \frac{n\pi z}{H} \cos m\theta \, dz d\theta$$

For $m = 0$ and $n \neq 0$, we get

$$A_{0n} = \frac{2}{n\pi^2 I_0'(n\pi a/H)} \int_0^{\pi} \int_0^{H} \beta(\theta, z) \cos \frac{n\pi z}{H} \, dz d\theta$$

For $m \neq 0$ and $n = 0$, we get

$$A_{m0} = \frac{2}{m\pi H a^{m-1}} \int_0^{\pi} \int_0^{H} \beta(\theta, z) \cos m\theta \, dz d\theta$$

Our boundary condition gives us no requirement on A_{00}. This makes sense because only derivatives have been specified in the problem, so the solution can only be unique up to a constant.

Math 241 (Spring 2019) Homework 9 Solutions

April 18, 2019

Problem 7.9.3(c). By separating variables $u(r,\theta,z,t) = f(r)q(\theta)g(z)h(t)$ one gets the ODEs

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\mu r^2 - \omega)f = 0 \qquad \frac{d^2q}{d\theta^2} = -\omega g \qquad \frac{d^2g}{dz^2} = -(\lambda-\mu)g \qquad \frac{dh}{dt} = -\lambda k h.$$
$$f'(a) = 0,\ |f(0)| < \infty \qquad q'(0) = q'(\pi/2) = 0 \qquad g(0) = g(H) = 0$$

After solving these ODEs one gets the general solution

$$u(r,\theta,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{mnl} J_{2m-1}(\sqrt{\mu_{mn}}r)\cos((2m-1)\theta)\sin\left(\frac{l\pi z}{H}\right)e^{-(\mu_{mn}+(l\pi/H)^2)kt}$$

where

$$\mu_{mn} = \left(\frac{y_{mn}}{a}\right)^2,\ \ y_{mn} \text{ are the zeros to } J'_m(z).$$

The coefficients can be solved using the initial condition:

$$f(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} A_{mnl} J_{2m-1}(\sqrt{\mu_{mn}}r)\cos((2m-1)\theta)\sin\left(\frac{l\pi z}{H}\right),$$

so

$$A_{mnl} = \frac{\int_0^H \int_0^{\pi/2} \int_0^a f(r,\theta,z) J_{2m-1}(\sqrt{\mu_{mn}}r)\cos((2m-1)\theta)\sin\left(\frac{l\pi z}{H}\right) r\,dr d\theta dz}{\int_0^H \int_0^{\pi/2} \int_0^a J^2_{2m-1}(\sqrt{\mu_{mn}}r)\cos^2((2m-1)\theta)\sin^2\left(\frac{l\pi z}{H}\right) r\,dr d\theta dz}$$

Finally, note that

$$\lim_{t\to\infty} u(r,\theta,z,t) = 0.$$

Problem 7.9.4(b). Since the initial condition is independent of θ we do not need to consider this (see Problem 7.10.1(c) below to see why heuristically). By the same steps as in the previous homework

$$u(r,\theta,z,0) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} A_{nj}\cos\left(\frac{n\pi z}{H}\right) J_0(\sqrt{\lambda_{0j}}r)e^{-\lambda_{0j}kt},$$

where

$$\lambda_{0j} = \left(\frac{y_{0n}}{a}\right)^2 + \left(\frac{n\pi}{H}\right)^2$$

and y_{0n} are the zeros of $J'_0(z)$. Using the initial condition

$$f(r,z) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} A_{nj}\cos\left(\frac{n\pi z}{H}\right) J_0(\sqrt{\lambda_{0j}}r)$$

and by orthogonality the coefficients are

$$A_{nj} = \frac{\int_0^{2\pi}\int_0^a f(r,z)\cos\left(\frac{n\pi z}{H}\right) J_0(\sqrt{\lambda_{0j}}r) r\,drdz}{\int_0^{2\pi}\int_0^a \cos^2\left(\frac{n\pi z}{H}\right) J_0^2(\sqrt{\lambda_{0j}}r) r\,drdz}.$$

Problem 7.10.1(c). By the separation of variables $u(\rho,\theta,\phi,t) = f(\rho)q(\theta)g(\phi)h(t)$ done in Section 7.10, one gets the following:

$$\frac{d}{d\rho}\left(\rho^2\frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f = 0 \qquad \frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0 \qquad \frac{d^2h}{dt^2} = -\lambda c^2 h$$

$$f(a) = 0,\ |f(0)| < \infty \qquad |g(0)| < \infty \qquad h'(0) = 0$$

and q satisfies the periodic ODE with eigenvalues m^2 and eigenfunctions $\cos m\theta$ and $\sin m\theta$ for $m = 0, 1, 2, \ldots$.

Note that g satisfies the Legendre ODE, and f satisfies the spherical Bessel's ODE. Thus

$$g(\phi) = P_n^m(\cos\phi),\quad n \geq m,$$
$$f(\rho) = c_1\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda}\rho).$$

The condition $f(a) = 0$ implies that

$$\lambda = \lambda_{n,k} = \left(\frac{z_{n+1/2,k}}{a}\right)^2$$

where $z_{n+1/2,k}$ are the solutions to the spherical Bessel equation. As for h, the usual considerations tells us that

$$h(t) = c_1\cos\left(c\sqrt{\lambda}t\right).$$

Thus the general solution is

$$u(\rho,\theta,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{k=1}^{\infty}\cos\left(c\sqrt{\lambda_{nk}}t\right)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nk}}\rho)(A_{nmk}\cos m\theta + B_{nmk}\sin m\theta)P_n^m(\cos\phi).$$

The initial condition tells us that

$$F(\rho,\phi) = u(\rho,\theta,\phi,0) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{k=1}^{\infty}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nk}}\rho)(A_{nmk}\cos m\theta + B_{nmk}\sin m\theta)P_n^m(\cos\phi).$$

Since F does not depend on θ we can assume $A_{nmk} = B_{nmk} =$ if $m \neq 0$. If $m = 0$ then $B_{n0k}\sin 0\theta = 0$ so we just need to compute A_{n0k}. By orthogonality the answer is

$$u(\rho,\theta,\phi,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}A_{n0k}\cos\left(c\sqrt{\lambda_{nk}}t\right)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nk}}\rho)P_n^0(\cos\phi)$$

with

$$A_{n0k} = \frac{\int_0^\pi\int_0^a F(\rho,\phi)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nk}\rho})P_n^m(\cos\phi)\rho^2\sin\phi\,d\rho d\phi}{\int_0^\pi\int_0^a(\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{nk}\rho}))^2(P_n^m(\cos\phi))^2\rho^2\sin\phi\,d\rho d\phi}.$$

Problem 7.10.2(c). Separate variables $u(\rho,\theta,\phi,t) = f(\rho)q(\theta)g(\phi)h(t)$, and get the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho^2\frac{\mathrm{d}f}{\mathrm{d}\rho}\right] + (\lambda\rho^2 - \mu)f = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\sin\phi\frac{\mathrm{d}g}{\mathrm{d}\phi}\right] + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0$$
$$\frac{\mathrm{d}h}{\mathrm{d}t} = -k\lambda h(t)$$

The $q(\theta)$ equation gives us the m^2 eigenvalue and eigenfunctions of $\cos m\theta$ and $\sin m\theta$. The ϕ equation fixes eigenvalues $\mu_{mn} = n(n+1)$ with eigenfunctions $g(\phi) = P_n^m(\cos\phi)$ (only for $n > m$).The ρ equation has solution $f(\rho) = \rho^{-1/2}J_{n+1/2}(\sqrt{\lambda}\rho)$ and fixes eigenvalues λ_{mnj} through the condition $J_{n+1/2}(\sqrt{\lambda}a = 0$. Finally, the time equation has the same exponential solution as always, and the general solution is

$$u(\rho,\theta,\phi,t)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{j=1}^{\infty}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{mnj}}\rho)(A_{mnj}\cos m\theta+B_{mnj}\sin m\theta)P_n^m(\cos\phi)e^{-k\lambda_{mnj}t}$$

The form of the inital condition eliminates all of the B_{mnj}'s, and leaves only the A_{1nj}'s, so we get

$$u(\rho,\theta,\phi,t)=\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}A_{1nj}\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{1nj}}\rho)\cos(\theta)P_n^1(\cos\phi)e^{-k\lambda_{1nj}t}$$

And plugging in the initial condition gives that

$$A_{1nj}=\frac{\int_0^\pi\int_0^a F(\rho,\phi)\rho^{-1/2}J_{n+1/2}(\sqrt{\lambda_{1nj}}\rho)P_n^1(\cos\phi)\rho^2\sin\phi\,d\rho d\phi}{\int_0^\pi\int_0^a \rho^{-1}J_{n+1/2}^2(\sqrt{\lambda_{1nj}}\rho)(P_n^1(\cos\phi))^2\rho^2\sin\phi\,d\rho d\phi}$$

Problem 7.10.10(a). As shown in example 7.10.6, the radial equation gives solutions ρ^n and ρ^{-n-1}. Outside a sphere, we reject the ρ^n solution, and we get a general solution that looks like

$$u(\rho,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\rho^{-n-1}\left[A_{nm}\cos m\theta+B_{nm}\sin m\theta\right]P_n^m(\cos\phi)$$

Our boundary condition tells us that

$$F(\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}a^{-n-1}\left[A_{nm}\cos m\theta+B_{nm}\sin m\theta\right]P_n^m(\cos\phi)$$

And hence we get that

$$A_{nm}=a^{n+1}\frac{\int_0^\pi\int_{-\pi}^\pi F(\theta,\phi)\cos m\theta P_n^m(\cos\phi)\sin\phi\,d\phi d\theta}{\int_0^\pi\int_{-\pi}^\pi \cos^2 m\theta(P_n^m(\cos\phi))^2\sin\phi\,d\phi d\theta}$$

$$B_{nm}=a^{n+1}\frac{\int_0^\pi\int_{-\pi}^\pi F(\theta,\phi)\sin m\theta P_n^m(\cos\phi)\sin\phi\,d\phi d\theta}{\int_0^\pi\int_{-\pi}^\pi \sin^2 m\theta(P_n^m(\cos\phi))^2\sin\phi\,d\phi d\theta}$$

Problem 7.10.11. Separate variables; the θ equation now gives eigenvalues of $(2m)^2$ and eigenfunctions $\sin 2m\theta$. The ϕ equation gives $P_n^{2m}(\cos\phi)$ and the radial equation gives only ρ^n since we are inside the sphere. So,

$$u(\rho,\theta,\phi)=\sum_{m=1}^{\infty}\sum_{n=2m}^{\infty}B_{nm}\rho^n\sin(2m\theta)P_n^{2m}(\cos\phi)$$

$$B_{nm}=a^{-n}\frac{\int_0^\pi\int_{-\pi}^\pi F(\theta,\phi)\sin(2m\theta)P_n^{2m}(\cos\phi)\sin\phi\,d\phi d\theta}{\int_0^\pi\int_{-\pi}^\pi \sin^2(2m\theta)(P_n^{2m}(\cos\phi))^2\sin\phi\,d\phi d\theta}$$

Math 241 (Spring 2019) Homework 10 Solutions

Last Homework, Hooray!

April 25, 2019

Problem 8.2.1(d). We solve for the equilibrium $u_E(x)$:

$$0 = k\frac{d^2u_E}{dx^2} + k, \qquad u(0) = A, \;\; u(L) = B.$$

This gives us

$$u_E(x) = -\frac{x^2}{2} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A.$$

Now we consider $v(x,t) = u(x,t) - u_E(x)$ with corresponding PDE

$$\frac{dv}{dt} = k\frac{d^2v}{dx^2}, \qquad v(0,t) = 0, \;\; v(L,t) = 0, \;\; v(x,0) = f(x) - u_E(x).$$

By doing computations exactly like in Section 8.2, one gets

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},$$
$$A_n = \frac{2}{L}\left(f(x) - u_E(x)\right)\sin\frac{n\pi x}{L}\,dx.$$

Then $u(x,t) = v(x,t) + u_E(x)$, where $u_E(x)$ and $v(x,t)$ are as above.

Problem 8.2.3. We solve for the equilibrium $u_E(r)$:

$$0 = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + Q(r), \qquad u(a) = T.$$

By the fundamental theorem of calculus,

$$r\frac{\partial u_E}{\partial r} = -\int_0^r \frac{s}{k}Q(s)\,ds$$

and so

$$T - u_E(r) = u_E(a) - u_E(r) = -\int_r^a \frac{1}{u}\int_0^u \frac{s}{k}Q(s)\,dsdu.$$

Hence

$$u_E(r) = T + \int_r^a \frac{1}{u}\int_0^u \frac{s}{k}Q(s)\,dsdu.$$

Now we consider $v(r,t) = u(r,t) - u_E(r)$ with corresponding PDE

$$\frac{dv}{dt} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right), \qquad v(r,0) = f(r) - u_E(r), \;\; v(a,t) = 0.$$

This is a standard Bessel's PDE with $m = 0$, so one gets

$$v(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{z_{0n}}{a} r\right) e^{-k(z_{0n}/a)^2 t},$$
$$A_n = \frac{\int_0^a (f(r) - u_E(r))\, J_0\left(\frac{z_{0n}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}.$$

Then $u(r,t) = v(r,t) + u_E(r)$, where $u_E(r)$ and $v(r,t)$ are as above.

Problem 8.3.1(d). We pick the linear function

$$w(x) = A - \frac{xA}{L},$$

since this satisfies $w(0) = A$ and $w(L) = 0$. Now we consider $v(x,t) = u(x,t) - w(x)$ with corresponding PDE

$$\frac{dv}{dt} = k\frac{d^2v}{dx^2} + Q(x,t), \qquad v(0,t) = 0,\ \ v(L,t) = 0,\ \ v(x,0) = f(x) - w(x).$$

The eigenfunctions are $\sin n\pi x/L$ for positive integers n. By the method of eigenfunction expansion we write

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi x}{L}$$

and solve for $A_n(t)$. To do this we can use Equations (8.3.7) and (8.3.9) and (8.3.10) to see that

$$A_n(t) = A_n(0)e^{-(n\pi/L)^2 kt} + e^{-(n\pi/L)^2 kt}\int_0^t q_n(\tau)e^{(n\pi/L)^2 kt}\, d\tau,$$
$$q_n(\tau) = \frac{\int_0^L Q(x,\tau)\sin(n\pi x/L)\, dx}{\int_0^L \sin^2(n\pi x/L)\, dx},$$
$$A_n(0) = \frac{\int_0^L (f(x) - w(x))\sin(n\pi x/L)\, dx}{\int_0^L \sin^2(n\pi x/L)\, dx}.$$

Then $u(x,t) = v(x,t) + w(x)$, where $w(x)$ and $v(x,t)$ are as above.

Problem 8.3.6. We pick the linear function

$$w(x) = 1 - \frac{x}{\pi},$$

since this satisfies $w(0) = 1$ and $w(\pi) = 0$. Now we consider $v(x,t) = u(x,t) - w(x)$ with corresponding PDE

$$\frac{dv}{dt} = \frac{d^2v}{dx^2} + e^{-2t}\sin 5x, \qquad v(0,t) = 0,\ \ v(L,t) = 0,\ \ v(x,0) = -w(x).$$

The eigenfunctions are $\sin nx$ for positive integers n. By the method of eigenfunction expansion we write

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin nx$$

and solve for $A_n(t)$. To do this we can use Equations (8.3.7) and (8.3.9) and (8.3.10) to see that

$$A_n(t) = \begin{cases} A_n(0)e^{-n^2 t} & \text{if } n \neq 5, \\ A_5(0)e^{-25t} + \frac{e^{-25t}}{23}\left(e^{23t} - 1\right) & \text{if } n = 5, \end{cases}$$
$$A_n(0) = -\frac{\int_0^\pi w(x)\sin nx\, dx}{\int_0^\pi \sin^2 nx\, dx} = -\frac{2}{n\pi}.$$

Then $u(x,t) = v(x,t) + w(x)$, where $w(x)$ and $v(x,t)$ are as above.

Problem 8.3.7. Let $v(x,t) = u(x,t) - \frac{xt}{L}$; then our PDE becomes

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{x}{L}$$

with the boundary conditions $v(0,t) = v(L,t) = 0$ and initial condition $v(x,0) = 0$. We can solve this using the method of eigenfunction expansion:

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t} \int_0^L \overline{q_n}(\tau) e^{k(n\pi/L)^2 \tau}\, d\tau$$

Note that the initial condition tells us that all $a_n(0) = 0$, and the form of $\overline{Q}$ gives us that

$$\overline{q_n}(\tau) = \frac{2}{L^2} \int_0^L x \sin \frac{n\pi x}{L}\, dx$$

$$\overline{q_n}(\tau) = \frac{2}{\pi n}(-1)^{n+1}$$

So, we get together

$$a_n(t) = \frac{2kL^2(-1)^{n+1}}{(n\pi)^3}$$

And the solution to the original problem is $u(x,t) = v(x,t) + \frac{xt}{L}$.

Problem 8.6.3(c). Split up $u = u_1 + u_2$, with u_l solving Laplace's equation with the inhomogeneous boundary condition, and u_2 solving Poisson's equation with a homogeneous boundary condition. First, we can solve for u_1 easily; the solution is

$$u_1 = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta$$

For u_2, we can solve the problem using two-dimensional eigenfunctions; the two-dimensional eigenfunctions of the Laplacian on the inside of the disk are

$$\phi_{mn} = J_m\left(z_{mn}\frac{r}{a}\right) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\}$$

With z_{mn} being the nth zero of J_m, the eigenvalues are $\left(\frac{z_{mn}}{a}\right)^2$. Hence, we get

$$u_2(r,\theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(z_{mn}\frac{r}{a}\right)(a_{mn}\cos m\theta + b_{mn} \sin m\theta)$$

$$a_{mn} = -\frac{a^2}{z_{mn}^2} \frac{\int_{-\pi}^{\pi} \int_0^a Q J_m\left(z_{mn}\frac{r}{a}\right) \cos m\theta\, r\, dr d\theta}{\int_{-\pi}^{\pi} \int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) \cos^2 m\theta\, r\, dr d\theta}$$

$$b_{mn} = -\frac{a^2}{z_{mn}^2} \frac{\int_{-\pi}^{\pi} \int_0^a Q J_m\left(z_{mn}\frac{r}{a}\right) \sin m\theta\, r\, dr d\theta}{\int_{-\pi}^{\pi} \int_0^a J_m^2\left(z_{mn}\frac{r}{a}\right) \sin^2 m\theta\, r\, dr d\theta}$$

Problem 8.6.6. Split it up again: let $u = u_1 + u_2$, with u_1 solving Laplace's equation with inhomogeneous boundary conditions, while u_2 solves Poisson's equation with all homogeneous boundary conditions. Then, the solution for u_1 looks like

$$u_1(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx) \sinh ny$$

$$b_n = \frac{2}{\pi \sinh nL} \int_0^{\pi} f(x) \sin nx \, dx$$

For u_2, the two-dimensional eigenfunctions are

$$\phi_{mn}(x,y) = \sin(nx) \sin \frac{m\pi y}{L}$$

and corresponding eigenvalue $\lambda_{mn} = n^2 + \left(\frac{m\pi}{L}\right)^2$.

So,

$$u_2(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin(nx) \sin \frac{m\pi y}{L}$$

$$a_{mn} = -\frac{1}{n^2 + (m\pi/L)^2} \frac{4}{\pi L} \int_0^L \int_0^{\pi} e^{2y} \sin x \sin nx \sin \frac{m\pi y}{L} \, dxdy$$

Problem 8.6.7. The eigenfunctions of the Laplacian inside this cube are given by

$$\phi_{mnj} = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \sin \frac{j\pi z}{W}$$

with eigenvalue

$$\lambda_{mnj} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{j\pi}{W}\right)^2$$

Write the solution as

$$u(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} b_{mnj} \phi_{mnj}(x,y,z)$$

Plugging this back into Poisson's equation gives us that

$$Q(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} -b_{mnj} \lambda_{mnj} \phi_{mnj}(x,y,z)$$

Which gives us that

$$b_{mnj} = \frac{-1}{\lambda_{mnj}} \frac{\iiint Q \phi_{mnj} \, dV}{\iiint \phi_{mnj}^2 \, dV}$$

$$b_{mnj} = \frac{-1}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{j\pi}{W}\right)^2} \frac{8}{LHW} \int_0^W \int_0^H \int_0^L Q(x,y,z) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \sin \frac{j\pi z}{W} \, dxdydz$$

Math 241 (Spring 2019) Homework 11 Solutions

Optional Homework

May 1, 2019

Problem 10.2.1. Start with

$$u(x,t) = \int_0^\infty \left[A(\omega)\cos\omega x e^{-k\omega^2 t} + B(\omega)\sin(\omega x)e^{-k\omega^2 t}\right] d\omega$$

$$u(x,t) = \int_0^\infty A(\omega)\frac{e^{i\omega x} + e^{-i\omega x}}{2} e^{-k\omega^2 t}\, d\omega + \int_0^\infty B(\omega)\frac{e^{i\omega x} - e^{-i\omega x}}{2i} e^{-k\omega^2 t}\, d\omega$$

$$u(x,t) = \int_0^\infty \frac{A(\omega) + iB(\omega)}{2} e^{-i\omega x} e^{-k\omega^2 t}\, d\omega + \int_0^\infty \frac{A(\omega) - iB(\omega)}{2} e^{i\omega x} e^{-k\omega^2 t}\, d\omega$$

$$u(x,t) = \int_{-\infty}^\infty C(\omega) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega$$

with

$$C(\omega) = \begin{cases} \frac{A(\omega)+iB(\omega)}{2}, & \omega > 0 \\ \frac{A(\omega)-iB(\omega)}{2}, & \omega < 0 \end{cases}$$

From which it is clear that $C(-\omega) = \overline{C(\omega)}$, if $A(\omega)$ and $B(\omega)$ are real.

Problem 10.2.2. To show $u(x,t)$ is real, we need to show $\overline{u(x,t)} = u(x,t)$.

$$\begin{aligned} \overline{u(x,t)} &= \overline{\int_{-\infty}^\infty c(\omega) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega} \\ &= \int_{-\infty}^\infty \overline{c(\omega)} e^{i\omega t} e^{-k\omega^2 t}\, d\omega \\ &= \int_{-\infty}^\infty c(-\omega) e^{i\omega t} e^{-k\omega^2 t}\, d\omega \end{aligned}$$

Making a change of variables $\omega \to -\omega$, we get

$$\begin{aligned} &= \int_{-\infty}^\infty c(\omega) e^{-i\omega t} e^{-k\omega^2 t}\, d\omega \\ &= u(x,t) \end{aligned}$$

Problem 10.3.1.

$$\begin{aligned} \mathcal{F}[c_1 f + c_2 g] &= \frac{1}{2\pi}\int_{-\infty}^\infty [c_1 f + c_2 g] e^{i\omega x}\, dx \\ &= \frac{1}{2\pi}\left[c_1 \int_{-\infty}^\infty f e^{i\omega x}\, dx + c_2 \int_{-\infty}^\infty g e^{i\omega x}\, dx\right] \\ &= c_1 F(\omega) + c_2 G(\omega) \end{aligned}$$

Problem 10.4.3. (a) Let $U(\omega, t)$ be the Fourier transform of $u(x, t)$. Then U satisfies

$$\frac{\partial U}{\partial t} = -k\omega^2 U - ic\omega U$$

$$U(\omega, 0) = F(\omega)$$

We can solve the ODE:

$$U(\omega, t) = F(\omega)e^{-k\omega^2 t}e^{-ic\omega t}$$

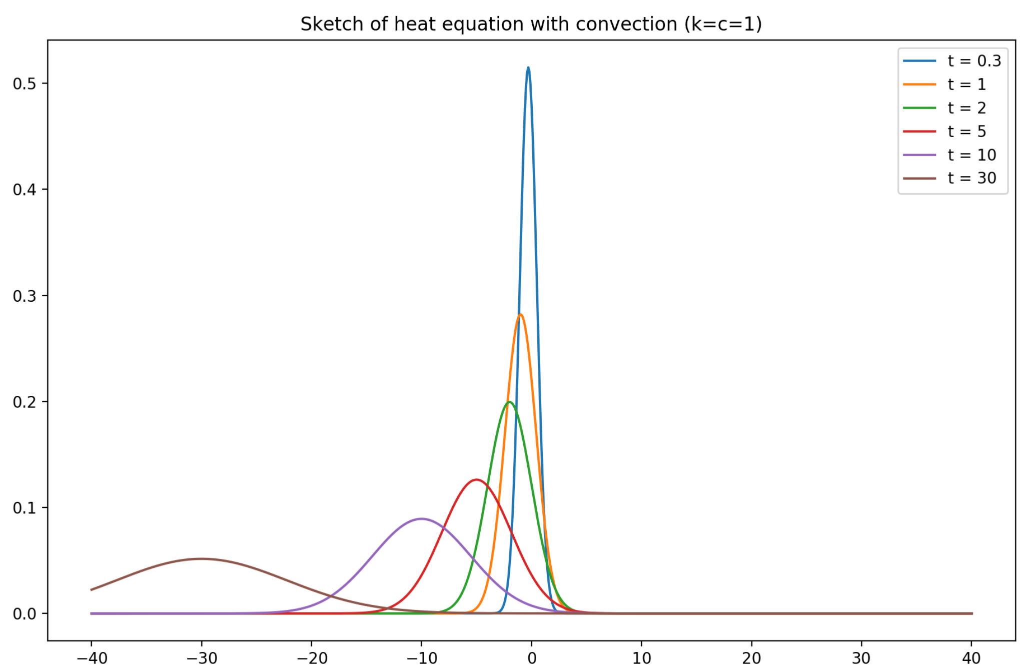
The inverse Fourier transform of $e^{-k\omega^2 t}$ is $\sqrt{\frac{\pi}{kt}}e^{-x^2/4kt}$, so the inverse Fourier transform of $F(\omega)e^{-k\omega^2 t}$ is (by the convolution theorem)

$$\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4kt}}\,dy$$

Then, by the shift theorem, $u(x, t)$ is this shifted by $-ct$:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x+ct-y)^2}{4kt}}\,dy$$

(b)



Problem 10.4.4. (a) Let $U(\omega, t)$ be the Fourier transform of $u(x, t)$. Then we get

$$\frac{\partial U}{\partial t} = -k\omega^2 U - \gamma U$$

$$U(\omega, 0) = F(\omega)$$

Which has the solution

$$U(\omega, t) = F(\omega)e^{-(k\omega^2+\gamma)t}$$

By the convolution theorem, we get

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\gamma t}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4kt}\,dy$$

(b) A simplifying transformation would be to let $v(x,t) = e^{\gamma t}u(x,t)$, then you would get the regular diffusion equation in v.

Problem 10.4.7(a)(b). (a) Let $U(\omega,t)$ be the Foruier transform of $u(x,t)$. Then we get

$$\frac{\partial U}{\partial t} = ik\omega^3 U$$

$$U(\omega,0) = F(\omega)$$

Which has the solution

$$U(\omega,t) = F(\omega)e^{ik\omega^3 t}$$

Hence,

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega)e^{ik\omega^3 t}e^{-i\omega x}\,d\omega$$

If $g(x)$ is the inverse Fourier transform of $e^{ik\omega^3 t}$, then we get (by the convolution theorem)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

$$g(x) = \int_{-\infty}^{\infty} e^{ik\omega^3 t}e^{-i\omega x}\,d\omega$$

Problem 10.6.3. Since y is the variable which has two homogeneous boundary conditions (at infinity), we Fourier transform $u(x,y)$ in y to get $U(x,\omega)$. Then, we get

$$\frac{\partial^2 U}{\partial x^2} - \omega^2 U = 0$$

As shown in the book, this means that U takes the form

$$U(x,\omega) = G(\omega)e^{-|\omega|y}$$

where $G(\omega)$ is the Fourier transform of $g(y)$. Also shown in the book is that the inverse Fourier transform of $e^{-|\omega|y}$ is $\frac{2x}{x^2+y^2}$ (they have done it with x and y switched). So,

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\overline{y})\frac{2x}{x^2+(y-\overline{y})^2}\,d\overline{y}$$

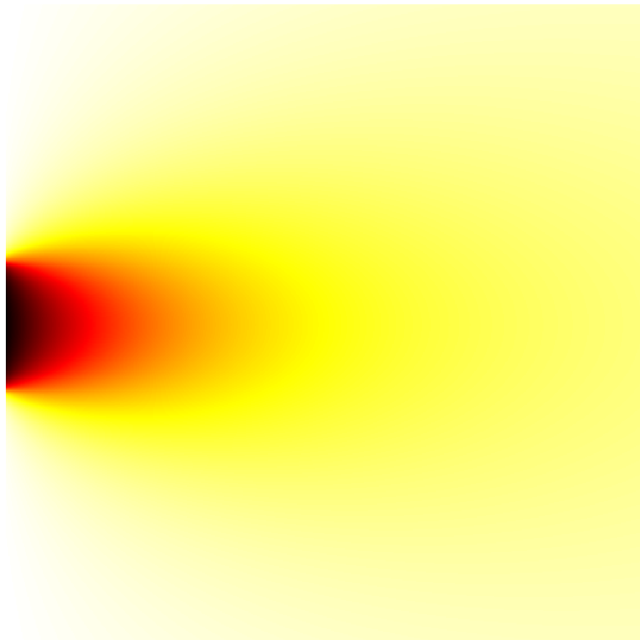
(b) This form for $g(y)$ just changes the bounds of the integral:

$$u(x,y) = \frac{x}{\pi}\int_{-1}^{1} \frac{1}{x^2+(y-\overline{y})^2}\,d\overline{y}$$

$$u(x,y) = \frac{1}{\pi}\left[\arctan\left(\frac{y-\overline{y}}{x}\right)\right]\Big|_{-1}^{1}$$

$$u(x,y) = \frac{1}{\pi}\left[\arctan\left(\frac{y-1}{x}\right) - \arctan\left(\frac{y+1}{x}\right)\right]$$

Here's a picture (it looks plausible at least, which is always a good sign!)



Problem 10.6.18. Let $U(\omega,t)$ be the Fourier transform of $u(x,t)$. Then, we get

$$\frac{\partial^2 U}{\partial t^2} = -c^2\omega^2 U$$

$$U(\omega,0)=0\ ;\ \frac{\partial U}{\partial t}(\omega,0)=G(\omega)$$

This has the solution

$$U(\omega,t)=a(\omega)\cos(\omega ct)+b(\omega)\sin(\omega ct)$$

The first initial condition means that $a(\omega)=0$, and the second tells us that $b(\omega)=\frac{G(\omega)}{\omega c}$. So,

$$U(\omega,t)=G(\omega)\frac{\sin\omega ct}{c\omega}$$

From the table, we know that the inverse Fourier transform of $\frac{\sin\omega ct}{\omega c}$ is

$$\frac{\pi}{c}\begin{cases}0, & |x|>ct\\ 1, & |x|<ct\end{cases}$$

Hence,

$$u(x,t)=\frac{1}{2c}\int_{-\infty}^{\infty} g(y)\begin{cases}0, & |x-y|>ct\\ 1, & |x-y|<ct\end{cases}dy$$

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$