The Satake isomorphism (2017-08-01)

The goal is to understand the Satake isomorphism, following the article by Cartier in the Corvallis volumes. We have the usual setup:

- F is a (nonarchimedean) local field.
- G is a connected reductive group over F.
- A is a maximal split in G.
- M is the centralizer of A in G.
- $N(A)$ is the normalizer of A in G.
- $W = N(A)/M$ is the Weyl group.
- P is a Borel subgroup, so $P = MN$ with N the unipotent radical.
- K is a "special" maximal compact of G, so it satisfies things like the Iwasawa decomposition $G = PK$ $(= MNK)$.
- $X^*(H) = \mathrm{Hom}_{k\text{-alg}}(H, \mathbb{G}_m)$ is the character group of $H \subset G$.
- $X_*(H) = \mathrm{Hom}_{\mathbb{Z}}(X^*(H), \mathbb{Z})$ is the cocharacter group of $H \subset G$.

We write ${}^\circ M = M \cap K$. Here is another characterization. Define $\mathrm{ord}_M : M \to X_*(M)$ by $\mathrm{ord}_M(m)(f) = \mathrm{ord}_F f(m)$. Then ${}^\circ M$ is the kernel of ord_M, i.e.

$$1 \to {}^\circ M \to M \xrightarrow{\mathrm{ord}_M} \Lambda =: \mathrm{ord}_M(M) \to 1.$$

What we want To understand the Hecke algebra $\mathcal{H}(G,K)$.

↳ Elements of $\mathcal{H}(G,K)$ are compactly supported functions $f : G \to \mathbb{C}$ that are K-biinvariant, with pointwise addition and convolution as multiplication: $f * g(m) = \int_{G} f(x)\, g(x^{-1}m)\, dm$.

Define $\mathcal{H}(M, {}^\circ M)$ similarly. Notice that a $\mathbb{C}$-basis for $\mathcal{H}(M, {}^\circ M)$ is the set $\{ch_\lambda : \lambda \in \Lambda\}$, where ch_λ is the characteristic function of $ord_M^{-1}(\lambda)$ in M.

Easy to check: $ch_\lambda * ch_{\lambda'} = ch_{\lambda+\lambda'}$ if we choose dm so that $\int_{{}^\circ M} dm = 1$.

↳ The idea is the usual $\int_M \cdots = \int_{{}^\circ M} \int_{{}^\circ M \backslash M} \cdots$ ∘

Corollary: $\mathcal{H}(M, {}^\circ M) = \mathbb{C}[\Lambda]$, where $\mathbb{C}[\Lambda]$ is the algebra generated by $\{e^\lambda : \lambda \in \Lambda\}$. ∘

Let now $\mathfrak{n} = \mathrm{Lie}(N)$, and define, for $m \in M$, $\delta(m) := |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F$. Note that

$$\int_N f(mnm^{-1})\, dn = \delta(m)^{-1} \int_N f(n)\, dn. \quad \text{——} \quad (*)$$

Def: The Satake map $S : \mathcal{H}(G,K) \longrightarrow \mathcal{H}(M, {}^\circ M) = \mathbb{C}[\Lambda]$ is defined by

$$Sf(m) = \delta(m)^{\frac{1}{2}} \int_N f(mn)\, dn \overset{(*)}{=\!=} \delta(m)^{-\frac{1}{2}} \int_N f(nm)\, dn.$$

Clearly Sf is ${}^\circ M$-biinvariant and compactly supported.

Thm (Satake): S is an algebra isomorphism onto $\mathbb{C}[\Lambda]^W$, so $\mathcal{H}(G,K)$ is abelian.

↳ What is the action of W on Λ? Notice that we have the diagram

$$\begin{tikzcd} 1 \arrow[r] & {}^\circ A \arrow[r] \arrow[d, hook] & A \arrow[r, two heads, "ord_A"] \arrow[d, hook] & X_*(A) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & {}^\circ M \arrow[r] & M \arrow[r, "ord_M"] & X_*(M) \arrow[r] & 1 \end{tikzcd}$$

with $X_*(A) \subset \Lambda \subset X_*(M)$. Thus $N(A)$ acts on M, ${}^\circ M$, A, ${}^\circ A$ via conjugation, and W acts on $X_*(M)$ with Λ as an invariant subspace.

Remark. In Buzzard-Gee the required isomorphism is $\mathcal{H}(G,K) = \mathbb{C}[X_*(T_d)]^W$

Over there A is already k-split so $M = A$.

Remark. The Satake isomorphism is the p-adic version of the Harish-Chandra isomorphism $Z(\mathfrak{g}) \longrightarrow (S\mathfrak{h})^W$ for G a complex semisimple Lie group.

We now sketch the proof of the above theorem.

Step 1. Check that S is a homomorphism by observing that it is actually the composition of three algebra maps

$$\mathcal{H}(G,K) \xrightarrow{\alpha} \mathcal{H}(P) \xrightarrow{\beta} \mathcal{H}(M) \xrightarrow{\gamma} \mathcal{H}(M)$$

where α is just restriction

β is given by $\beta u(m) = \int_N u(mn)\, dn$

γ is given by $\gamma f(m) = f(m)\, \delta(m)^{\frac{1}{2}}$.

Step 2. Check that $S(\mathcal{H}(G,K))$ is contained in $\mathbb{C}[\Lambda]^W$.

$\hookrightarrow$ Use fact: $Sf(m) = \Delta(m)\, \delta(m)^{-\frac{1}{2}} \int_{G/A} f(gmg^{-1})\, dm$.

Another property of K is that $W = (N(A) \cap K)/{}^{\circ}M$. Hence we need to show that $Sf(xmx^{-1}) = Sf(m)$ for $m \in M$ and $x \in N(A) \cap K$. In fact it suffices to show it for elements $m \in M$ such that $m \longmapsto \det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)$ is (polynomial) nonzero.

$\boxed{\text{Invariance for } D(m)}$ In fact $D(xmx^{-1}) = D(m)$ for $x \in N(A)$. This follows because $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$ so

$$D(m)^2 = |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)|_F^2 \cdot |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F^{-1}$$

$$= |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)_F|\, |\det(\mathrm{Ad}_{\mathfrak{n}}(m^{-1}) - 1|_F = |\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1)|_F\, |\det(\mathrm{Ad}_{\mathfrak{n}^-}(m) - 1|_F$$

Thus $D(x) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{m}}(m) - 1)|_F^{\frac{1}{2}}$, so $D(xmx^{-1}) = D(m)$.

$\boxed{\text{Invariance for integral}}$ Note that $N(A) \cap K$ acts by inner automorphisms on G and A; so leaves invariant the measure on G/A. Let $m \in M$ be regular, $x \in N(A) \cap K$, and $f \in \mathcal{H}(G,K)$. Thus $f(xgx^{-1}) = f(g)$ for any $g \in G$, and

$$\int_{G/A} f(g(xmx^{-1})g^{-1})\, d\bar{g} = \int_{G/A} f((x^{-1}gx)\, m(x^{-1}gx)^{-1})\, d\bar{g} = \int_{G/A} f(gmg^{-1})\, d\bar{g}.$$

Step 3. Check that $S(\mathcal{H}(G,K)) \xrightarrow{\cong} \mathbb{C}[\Lambda]^W$.

↳ The idea is to find another basis of $\mathcal{H}(G,K)$ using the Cartan decomposition and show that the image of the basis is "upper triangular" with respect to the original basis ch_λ of $\mathbb{C}[\Lambda]$.

Here is an application. Let us try to determine all unitary algebra homomorphisms $\mathcal{H}(G,K) \to \mathbb{C}$. We do this by looking at the ones for $\mathcal{H}(M, {}^\circ M)$ and then passing over to $\mathcal{H}(G,K)$ via the Satake isomorphism. Since $\int_{{}^\circ M} dm = 1$ by assumption, the map $f \mapsto \int_M f(m)\chi(m)\, dm$ is a unitary homomorphism for all unramified $\chi: M \to \mathbb{C}^\times$, and all such unitary maps arises in this way.

Corollary: Any unitary $\mathcal{H}(G,K) \to \mathbb{C}^\times$ is of the form $w_\chi(f) = \int_M Sf(m)\chi(m)\, dm$ for an unramified χ. Moreover, $w_\chi = w_{\chi'}$ iff $\chi' = w \cdot \chi$ for some $w \in W$.

<u>Definition</u>: A <u>spherical function</u> of G with repect to K is a function

$\Gamma: G \to \mathbb{C}$ that is K-invariant, with $\Gamma(1)=1$, and such that

for any $f \in \mathcal{H}(G,K)$ there is a constant $\lambda(f)$ with $f * \Gamma = \Gamma * f = \lambda(f)\,\Gamma$.

Our goal now is to translate our results in terms of this language. Let χ be an unramified character of M, and define

$$\Phi_{K,\chi}(\overbrace{mnk}^{G=MNK}) = \chi(m)\,\delta^{\frac{1}{2}}(m) \quad \text{for } m \in M,\ n \in N,\ k \in K.$$

$$\Gamma_\chi(g) = \int_K \Phi_{K,\chi}(kg)\,dk, \quad \text{for } g \in G.$$

<u>Thm</u>: (a) The spherical functions are the functions Γ_χ.

(b) $\Gamma_\chi = \Gamma_{\chi'} \Leftrightarrow \chi = w\chi'$ (some $w \in W$).

↳ Computation for (b) in the notes.

↳ Spherical functions comes up in the Hecke theory for irreducible admissible representations and automorphic forms for GL_n, say.

(Godement's notes as a reference).

The last condition for spherical function is that it should be like an eigenfunction for the Hecke operator on $L^2_0(G_k \backslash G_{\mathbb{A}}, \omega)$.