Oral Exam Transcript

Yao-Rui

Committee: Ching-Li Chai (CL), David Harbater (DH), James Haglund (JH)

1 Syllabus

Major Topic: Number Theory

- **Valuation theory:** Local and global fields, extension of valuations, completions, weak and strong approximation, Hensel's lemma, Krasner's lemma, product formula, adeles and ideles.
- **Ramification theory:** Dedekind domains, prime decomposition and ramification, Dirichlet's unit theorem, finiteness of class number, difference and discriminant, Hermite's theorem, quadratic reciprocity.
- **Class field theory:** Artin reciprocity, statements of local and global class field theory.
- **L-functions:** Definitions of Dirichlet and Hecke and Artin L-functions, statements of class number formula and Cebotarev's density theorem, Dirichlet density.

References: John Cassels and Albrecht Frohlich, *Algebraic Number Theory*, Chapters I–VIII.

Minor Topic: Combinatorics

- **Enumeration:** Inversions and descents, permutations of multisets, partition numbers, the Twelvefold Way, inclusion and exclusion, rook polynomials and Ferrers boards, involutions and determinants.
- **Applications of symmetric functions:** The RSK algorithm, the Jacobi-Trudi formula and the Schur function, the Littlewood-Richardson rule, the Murnaghan-Nakayama rule.

Reference: Bruce Sagan, *The Symmetric Group*, Chapter 4; Richard Stanley, *Enumerative Combinatorics, Volume 1 (Second Edition)*, Chapters 1.1–1.5, 1.7–1.9, 2.1–2.4, 2.6–2.7.

2 Exam Synopsis

JH suggested the major-minor-major format, so the committee decided to go with it.

Major (Part 1)

DH: Find all extensions of $\mathbb{Q}$ such that the primes p satisfying $p \equiv 1 \pmod 5$ are totally split.

DH: Is 251 a square modulo 5? Is 5 a square modulo 251? Relate this to the splitting of the primes 251 and 5 in some two quadratic fields.

DH: Give an example of a number field K with its ring of integers not a PID. Why is it not a PID? Write an unramified extension H of K where all the primes of K are principal. Is H a PID?

DH: You wrote down the extension $\mathbb{Q}(\sqrt{-5}, \sqrt{-1})$ of $\mathbb{Q}(\sqrt{-5})$ and claimed that it is a degree two Galois extension. What is the general theorem you are using? What is the extension degree of the Hilbert class field? How do you prove that this extension is finite?

DH: Can you describe how one might construct an infinite degree unramified extension? (Hint: class field tower.) Can you give such an example?

CL: You said that 251 is totally split in $\mathbb{Q}(\sqrt{5})$. How do you construct a q-Weil number of slope 2/1 for 251 inside $\mathbb{Q}(\sqrt{5})$?

Minor

JH: What is a formula for the derangement number? Prove it using rook polynomials. What is the probability that a permutation of S_n has the derangement property as $n \to \infty$?

JH: Define the Schur function via the semi-standard Young tableau. Why is it symmetric?

JH: Write down the following polynomial:

$$\sum_{1 \le \sigma_1 \le \cdots \le \sigma_n \le n} x_{\sigma_1} \cdots x_{\sigma_n}.$$

Why is it a Schur-positive function? In fact this is a Schur function. Which one is it?

JH: Now write down the following polynomial:

$$\left(\sum_{i=1}^{n} x_i \right)^n.$$

Write this as a sum of Schur functions using the RSK algorithm. Which representation of S_n has this as its character?

JH: What is the image of the multiset permutation 3212 under the RSK algorithm? How can you tell the descent set of a multiset permutation using the RSK algorithm?

Major (Part 2)

CL: Let $K = \mathbb{Q}(\sqrt[3]{23}, \sqrt{-3})$ (DH: or $K = \mathbb{Q}$). Is K closed in $\mathbb{A}_K$? Is it discrete? Is it dense?

CL: Let K be as above. Is K closed in the finite adeles $\mathbb{A}_{K,f}$? Is it discrete? Is it dense?

CL: Let K be as above. Is $K^\times$ closed in the finite ideles $\mathbb{A}_{K,f}^\times$? Is it discrete? Is it dense?

CL: Let K be as above. Define its Dedekind ζ-function and its Artin L-function. Can you relate these two expressions by writing the ζ-function in terms of the L-function?

CL: State two main properties of the Artin L-function.

Length of exam: 100 minutes.

3 Post-Mortem

The exam committee was really nice and dropped many hints along the way when I got stuck. They did not seem to mind when I could not accurately state some non-syllabus material off the top of my head (for example, stating precisely the definition of q-Weil number and the induction property of the Artin L-function). They are really picky though about stating everything correctly for the things that you should know.

The oral exam only tests a fraction of the things you put on the syllabus (which is in turn a fraction of the things you prepared for the oral exam) since it is a timed exam. However, it seems that the committee asks problems to test you on the absolute basics, and they aren't too concerned if you don't answer the fancier things fully. In fact, the questions they asked during the exam were easier than the problems they gave me to prepare for it. A list of such problems will be written down in another document.