COMPLEX ANALYSIS CRASH COURSE

YAO-RUI

We present some complex analysis necessary to understand classical number-theoretic things like Riemann zeta, gamma function, elliptic functions, modular forms and so on.

1. Complex differentiation

Recall that the norm of a complex number $z = x + yi$ is $|z| = \sqrt{x^2 + y^2}$. This can be checked to satisfy positive definiteness, absolute scalability with respect to $\mathbb{C}$, and the triangle inequality.

Definition 1.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex function. This function has a *limit* A as z tends to z_0, written

$$\lim_{z \to z_0} f(z) = A,$$

if the following holds: for all $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$. The function f is said to be *continuous at* z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Recall that the conjugate of a complex number $z = x + yi$ is defined as $\overline{z} = x - yi$. We also define the real and complex parts of z as $\Re(z) = x$ and $\Im(z) = y$.

Proposition 1.2. *If* $\lim_{z \to z_0} f(z) = A$, *then* $\lim_{z \to z_0} \overline{f(z)} = \overline{A}$. *Consequently,* $\lim_{z \to z_0} f(z)$ *exists if and only if* $\lim_{z \to z_0} \Re(f(z)) = \Re(A)$ *and* $\lim_{z \to z_0} \Im(f(z)) = \Im(A)$.

Definition 1.3. The derivative of f at z_0 is defined to be

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

It exists if and only if the right hand limit exists. If $f'(z_0)$ exists for all $z_0 \in D$ for some open set $D \subset \mathbb{C}$, then we say f is *analytic*, or *holomorphic*, at D. If $f'(z_0)$ exists for all $z_0 \in \mathbb{C}$, then we say f is *entire*.

The usual limit and differentiation rules still hold. For example, differentiability implies continuity. In fact, we will see later on that if $f'(z_0)$ exists, then $f^{(n)}(z_0)$ exists for all $n \geq 1$, so once-differentiability implies smoothness in the complex case.

Proposition 1.4 (Cauchy-Riemann equations)**.** *Write* $f(z) = u(x, y) + iv(x, y)$. *If* f *is analytic at a point, then over there* u *and* v *satisfies*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are known as the Cauchy-Riemann equations. Conversely, if the first order partial derivatives of u *and* v *exists and are continuous, and furthermore satisfies the Cauchy-Riemann equations, then* f *is analytic.*

The proof is an easy computation. To prove the first claim, we differentiate f along the real and imaginary axes. This gives us

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$
$$f'(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} = -\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the two equalities gives us the Cauchy-Riemann equations. To prove the converse, note that

$$\lim_{h \to 0} (u(x+h, y) - u(x, y)) = h\frac{\partial}{\partial x}u(x, y)$$
$$\lim_{k \to 0} (u(x+h, y+k) - u(x+h, y)) = k\frac{\partial}{\partial y}u(x+h, y) = k\frac{\partial}{\partial y}u(x, y) + e_1,$$

where by continuity of the partial derivatives e_1 exists and can be chosen to tend more rapidly than h to zero. Therefore

$$u(x+h,y+k)-u(x,y)=h\frac{\partial u}{\partial x}+k\frac{\partial u}{\partial y}+\epsilon_1,$$

where ϵ_1 exists and again can be chosen to tend more rapidly than $h+ik$ to zero. By the same reasoning there exists ϵ_2 with the same property as ϵ_1 such that

$$v(x+h,y+k)-v(x,y)=h\frac{\partial v}{\partial x}+k\frac{\partial v}{\partial y}+\epsilon_2.$$

Consequently, by the Cauchy-Riemann equations, we can write

$$f(z+h+ik)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(h+ik)+\epsilon_1+i\epsilon_2,$$

so that

$$\lim_{h+ik\to 0}\frac{f(z+h+ik)-f(z)}{h+ik}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

This implies $f'(z)$ exists at all points in the domain, so f is analytic.

As a side remark, the Cauchy-Riemann equations are sometimes written more compactly as $\partial f/\partial\overline{z}=0$, where

$$\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)\qquad\text{and}\qquad\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right).$$

Notice that the Cauchy-Riemann equations already distinguishes complex analysis with analysis of real functions. For example, the function $z\mapsto\overline{z}$ is differentiable in the real sense as it is simply $(x,y)\mapsto(x,-y)$, but not in the complex sense as it fails to satisfy the Cauchy-Riemann equations.

Recall that the argument $\arg z$ of a complex number $z=x+yi$ is defined to be the angle from the positive x-axis to z such that $\tan(\arg z)=y/x$. The argument satisfies $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$, and by convention $\arg z\in(-\pi,\pi]$. Also recall that, by Euler's formula, every complex number $z=x+yi$ can be written in the form $z=re^{i\theta}=r(\sin\theta+i\cos\theta)$, where $r=\sqrt{x^2+y^2}$ and $\theta=\arg z+2\pi n$ for any integer n.

Corollary 1.5 (Polar Cauchy-Riemann equations)**.** *Let us write $f(z)=u(r,\theta)+iv(r,\theta)$ in polar coordinates. The Cauchy-Riemann equations can be written in polar coordinates as*

$$r\frac{\partial u}{\partial r}=\frac{\partial v}{\partial\theta}\qquad\text{and}\qquad\frac{\partial u}{\partial\theta}=-r\frac{\partial v}{\partial r}.$$

Further, the derivative of f can be expressed as

$$f'(z)=e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right).$$

The proof of this is left as an exercise on computation in polar coordinates.

2. Complex logarithm and trigonometric functions

Definition 2.1. The *(multi-valued) logarithm* is defined to be $\log z:=\ln r+i(\theta+2\pi n)$, where n ranges over the integers and $\theta=\arg z$. The *principal branch* of the logarithm is defined to be $\operatorname{Log} z:=\ln r+i\theta$.

The logarithm is well-defined if we restrict our domain of the argument to $(\alpha,\alpha+2\pi]$ for some real number α, and it satisfies $e^{\log z}=z$.

Definition 2.2. A *branch* of a multi-valued function f is the restriction of it to some domain such that it is well-defined and analytic over there. A curve C is a *branch cut* if its complement is a branch of f.

Example 2.3. The principal branch of the logarithm Log is analytic at $\mathbb{C}\setminus(-\infty,0]$ by the polar Cauchy-Riemann equations, and is not well-defined at $(-\infty,0]$ due to the periodicity of the complex exponential. Hence Log is branched at $\mathbb{C}\setminus(-\infty,0]$ with branch cut $(-\infty,0]$.

Similarly, $(\alpha,\alpha+2\pi]$ is a branch of log, with branch cut the ray $\{k\alpha : k\in\mathbb{R}_{\geq 0}\}$. We can also consider the various translations and composition of functions to get weird branches and branch cuts of log.

The logarithm satisfies all the usual properties as in the real case, as one might expect. We still have $\log(z_1z_2)=\log(z_1)+\log(z_2)$ and $\log(z_1^{-1})=-\log(z_1)$. The only difference is that we need to specify a branch for this to make sense, and the argument should be taken modulo 2π with respect to the branch cut.

In the future, we shall be most interested in using the principal branch of the logarithm. Further, the exponentiation of two complex numbers $z_1^{z_2}$ is understood as short-hand for

$$z_1^{z_2} := e^{z_2 \operatorname{Log} z_i}.$$

Notice then that complex exponentiation is also defined everywhere except at a ray outwards from the origin.

Definition 2.4. The *sine*, *cosine*, and *tangent* functions are defined as

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \tan z := \frac{\sin(z)}{\cos(z)}.$$

Their inverse counterparts are defined as

$$\csc z := \frac{1}{\sin(z)}, \qquad \sec z := \frac{1}{\cos(z)}, \qquad \cot z := \frac{1}{\tan(z)}.$$

Their hyperbolic counterparts are defined as

$$\sinh z := \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z := \frac{e^{z} + e^{-z}}{2}, \qquad \tanh z := \frac{\sinh(z)}{\cosh(z)}.$$

It is not hard to see that, just like the logarithm, they reduce to the usual cases in the real case using Euler's formula. They also satisfy all the usual trigonometric identities and differentiation relations, which we will not rewrite here. Let us note two that might be useful in the future.

Proposition 2.5. *Writing $z = x + yi$, we have the trigonometric identities*

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x),$$
$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x).$$

3. Line integrals and Cauchy's closed curve theorem

Definition 3.1. Suppose $f(t) = u(t) + iv(t)$ is a continuous function on an interval (a, b). Then we define

$$\int_a^b f(t)dt := \int_a^b u(t)dt + \int_a^b v(t)dt.$$

If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a continuous complex function, and $\gamma : [a, b] \longrightarrow \mathbb{C}$ is a piecewise differentiable arc on (a, b) with f defined on the image of γ, then its *line integral* is defined to be

$$\int_\gamma f = \int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Notice in the above definition of the complex integral we used proposition 1.2 to deduce that $u(t)$ and $v(t)$ must be continuous as well, so the integral is well-defined as continuous functions are integrable. The line integral satisfies several important fundamental properties.

Proposition 3.2. *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous complex function, and let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise differentiable arc with f defined on the image of γ.*

(a) The line integral is additive and linear in $\mathbb{C}$, i.e. if g is another continuous complex functions and $c \in \mathbb{C}$ then $\int_\gamma (f + g) = \int_\gamma f + \int_\gamma g$ and $\int_\gamma (cf) = c \int_\gamma f$.

(b) If we divide γ into a finite number of subarcs $\gamma_1 + \cdots + \gamma_n$, then $\int_\gamma f = \sum_i \int_{\gamma_i} f$.

(c) The line integral is invariant under a change of parameter, i.e. if $\tau : (\alpha, \beta) \longrightarrow (a, b)$ is an increasing piecewise differentiable function, then $\int_\gamma f = \int_{\gamma \circ \tau} f$.

(d) Let $-\gamma$ be the inverse arc defined by $-\gamma(t) := \gamma(a + b - t)$. Then $\int_{-\gamma} f = -\int_\gamma f$.

(e) Suppose $|f|$ is bounded by a real number M. Then $\left|\int_\gamma f\right| \leq M|\gamma|$, where $|\gamma|$ is the arc length of γ.

(f) Suppose $\{f_n\}$ is a sequence of continuous functions that converges uniformly to f on γ. Then

$$\int_\gamma f = \lim_{n \to \infty} \int_\gamma f_n.$$

(g) Suppose $\sum_{i=-\infty}^{\infty} a_i$ *is a complex series that converges. Then*

$$\int_\gamma f(z)\left(\sum_{i=-\infty}^{\infty} a_i\right) dz = \sum_{i=-\infty}^{\infty}\left(\int_\gamma f(z)a_i\right) dz.$$

(h) Suppose f *is the derivative of an analytic function* F*. Then* $\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$*.*

The proof of this proposition is a good exercise in calculus. We will do part 9c), leaving the remaining ones as an exercise. To prove part (c), we use the chain rule to see that

$$\begin{aligned}\int_{\gamma\circ\tau} f &= \int_\alpha^\beta f\left(\gamma(\tau(t))\right)\gamma'(\tau(t))\tau'(t)dt \\ &= \int_a^b f\left(\gamma(u)\right)\gamma'(u)du \\ &= \int_\gamma f.\end{aligned}$$

As for part (f), letting $\epsilon > 0$, by assumption there exists a positive integer N such that $|f(z) - f_n(z)| < \epsilon$ for all $n \geq N$ and all z lying on γ. Then part (e) tells us that

$$\left|\int_\gamma f - \int_\gamma f_n\right| < |f - f_n||\gamma| < \epsilon|\gamma|$$

for all $n \geq N$. This gives us what we want.

We now give some other definitions of line integrals that we will use sparingly.

Definition 3.3. The integral of γ with *respect to arc length* and *conjugation* are respectively written as

$$\int_\gamma f(z)|dz| := \int_\gamma f(\gamma(t))|\gamma'(t)|dt \qquad \text{and} \qquad \int_\gamma f(z)\overline{dz} := \overline{\int_\gamma \overline{f}}.$$

The line integral of f *with respect to* x, or y, are written respectively as

$$\int_\gamma f dx = \frac{1}{2}\left(\int_\gamma f dz + \int_\gamma f\overline{dz}\right) \qquad \text{and} \qquad \int_\gamma f dy = \frac{1}{2i}\left(\int_\gamma f dz - \int_\gamma f\overline{dz}\right).$$

The reason we introduced this integral is because we want to prove the main theorem of line integrals in the complex case, i.e. when a line integral only depend on its end points. This is answered as a corollary of the following theorem of two real variables. Recall that connected open subsets of $\mathbb{R}^n$ are path-connected.

Theorem 3.4. *Let* p *and* q *be continuous functions in two variables, and let* $\gamma : (a, b) \longrightarrow \mathbb{R}^2$ *be a piecewise differentiable arc with* p *and* q *defined on the image of* γ*. Also let* D *be a connected subspace of* $\mathbb{R}^2$ *containing* $\operatorname{im}\gamma$*. Then the line integral* $\int_\gamma pdx + qdy$*, defined in* D*, depends only on the end points of* γ *if and only if there exists a function* $U(x, y)$ *defined in* D *such that* $\partial U/\partial x = p$ *and* $\partial U/\partial y = q$*.*

The essential part of the proof of theorem 3.4 is the computation

$$\begin{aligned}\int_\gamma pdx + qdy &= \int_a^b \left(x'(\gamma(t))\frac{\partial U}{\partial x} + y'(\gamma(t))\frac{\partial U}{\partial y}\right) dt \\ &= \int_a^b \frac{d}{dt}U(x((\gamma(t)), y(\gamma(t))))dt \\ &= U(x(\gamma(b)), y(\gamma(b))) - U(x(\gamma(a)), y(\gamma(a))),\end{aligned}$$

where we have used the fundamental theorem of calculus in the last line.

Corollary 3.5. *The line integral* $\int_\gamma f$ *depends only on the end points of* γ *if and only if* f *is the derivative of an analytic function* F *in its domain.*

The proof of this corollary is simple given theorem 3.4. To prove the forward inclusion, the theorem above gives us an F with $\partial F/\partial x = f(z)$ and $\partial F/\partial y = if(z)$. This implies, after writing $F = U + iV$, that

$$\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = f(z) = -i\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y}.$$

which are the Cauchy-Riemann equations. Hence F is analytic by theorem 1.4 since f is continuous. The backward inclusion is immediate by the theorem above after identifying the complex plane $\mathbb{C}$ with the real plane $\mathbb{R}^2$, since $\int_\gamma f = \int_\gamma f dx + i f dy$.

As another application of theorem 3.4, we introduce harmonic functions in the following example.

Example 3.6 (Harmonic functions)**.** A real function $u(x, y)$ in two variables is *harmonic* if it is a twice continuously differentiable function that satisfies Laplace's equation, i.e. if $\Delta u := \partial^2 u/\partial x^2 + \partial^2 v/\partial y^2 = 0$. A *harmonic conjugate* of u is another harmonic function $v(x, y)$ such that $u + iv$ is an analytic function, or equivalently that u and v satisfies the Cauchy-Riemann equations. Note that a harmonic conjugate of u always exists by considering

$$\int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial t}(s,t)ds + \frac{\partial u}{\partial s}(s,t)dt.$$

In fact, theorem 3.4 tells us that we can integrate along any path that starts at the fixed point (x_0, y_0) and ends at (x, y).

Unlike the real case, there is a second main theorem of complex line integrals, called Cauchy's theorem, which we state below. We sometimes call it *Cauchy's theorem for closed curves* to avoid confusion with Cauchy's integral formula in the next section.

Theorem 3.7 (Cauchy)**.** *Let D be an open disk.*

(a) If f is analytic in D, then $\int_\gamma f = 0$ for every closed curve γ in D.

(b) Suppose f is analytic in D', where D' is obtained from D by omitting a finite number of points p_j. If f satisfies the condition $\lim_{z\to p_j}(z - p_j)f(z) = 0$ for all j, then $\int_\gamma f = 0$ for any closed curve γ in D'.

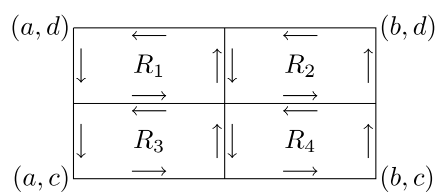
A partial converse to the first part of Cauchy's theorem, called Morera's theorem, will be proved in the next section after we look at Cauchy's integral formula. As a preliminary to the proof of Cauchy's theorem, we consider the case when D is a rectangle. In the next three lemmas, R will be the rectangle defined by $[a, b] \times [c, d] \subset \mathbb{R}^2$.

Lemma 3.8. *$\int_{\partial R}(az + b)dz = 0$ for any $a, b \in \mathbb{C}$.*

The above lemma is a simple exercise using corollary 3.5.

Lemma 3.9. *If f is analytic in R, then $\int_{\partial R} f = 0$.*

Proof. We use the method of bisection. Divide the rectangle R into four equal regions as shown.



We use the basic properties of integration (proposition 3.2) throughout our proof without mention. If we define $I(R_n) = \int_{\partial R_n} f$, then $I(R) = \sum_{i=1}^{4} I(R_i)$. Furthermore, we must have $|I(R_1)| \geq |I(R)|/4$ without loss of generality. By bisecting R_1 in the same way, and iterating, we have a sequence of nested rectangles $R \supset R_1 \supset R_5 \supset \cdots \supset R_{4n+1} \supset \cdots$, defined up to relabeling, such that

$$|I(R_{4n+1})| \geq \frac{|I(R)|}{4^{n+1}}.$$

Note that, if we write $|R|$ to be the perimeter of R and d_R to be the diagonal of R, then $|R_{4n+1}| = |R|/2^{n+1}$ and $d_{R_{4n+1}} = d/2^{n+1}$.

Clearly our chosen sequence of nested rectangles converge to a point $r \in R$. Hence, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - r| < \delta$, then

$$\left|\frac{f(z) - f(r)}{z - r} - f'(r)\right| < \epsilon.$$

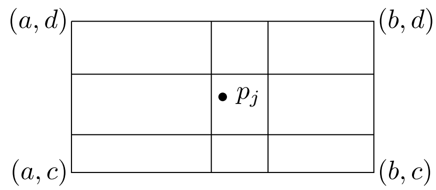
By lemma 3.8, $\int_{\partial R_{4n+1}} f(r)dz = \int_{\partial R_{4n+1}} (z-r)f'(r)dr = 0$. Hence the inequality tells us that

$$|I(R_{4n+1})| = \left| \int_{\partial R_{4n+1}} f(z) - f(r) - (z-r)f'(r) \right| < \frac{\epsilon d|R|}{2^{2n+2}},$$

implying that $|I(R)| < \epsilon d|R|$. Hence $|I(R)| = 0$ as ϵ is arbitrary, as desired. ☐

Lemma 3.10. *Let f be analytic on the set R', where R' is obtained from R by omitting a finite number of interior points p_j. If $\lim_{z \to p_j}(z - p_j)f(z) = 0$ for all j, then $\int_{\partial R} f = 0$.*

Proof. It suffices to consider the case of a single exceptional point p_j, since we can subdivide R' into this case (and use part (d) of proposition 3.2). Divide R' into nine possibly nonequal rectangles as shown, and call the center rectangle R_0.



Applying lemma 3.9 to all but the rectangle R_0, we see that $\int_{\partial R} f = \int_{\partial R_0} f$. By assumption, if $\epsilon > 0$ then we can choose R_0 to be a square and such that

$$|f(z)| < \frac{\epsilon}{|z - p_j|}$$

as z ranges over ∂R_0. Hence, as $|\partial R_0| < 8|z - p_j|$, we see that

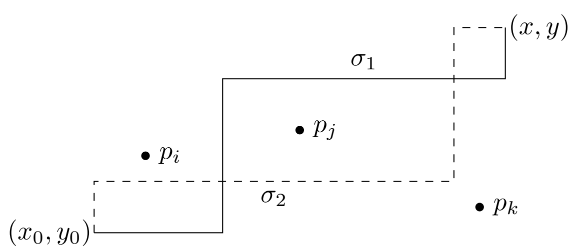
$$\left| \int_{\partial R_0} f \right| < \epsilon \int_{\partial R_0} \frac{1}{|z - p_j|} |dz| < 8\epsilon$$

This implies what we wanted, since ϵ was arbitrary. ☐

With this we are now ready to prove Cauchy's theorem for closed curves.

Proof of theorem 3.7. (a) Let $c = (x_0, y_0)$ be the center of the disk D. Then clearly any rectangle with a corner at c with diagonal inside D will lie inside D. Define the function $F(z) := \int_{\sigma_1} f(z)dz$, where σ_1 consists of two straight line segments connecting (x_0, y_0) to (x, y_0) to (x, y). We see immediately that $\partial F/\partial y = if(z)$, since $\int_{\sigma_1} f = \int_{\sigma_1} f dx + i f dy$. Lemma 3.9 then tells us that σ_1 in the definition of F can be replaced by the path σ_2, which consists of two straight line segments connecting (x_0, y_0) to (x_0, y) to (x, y). Hence $\partial F/\partial x = f(z)$. This implies F is analytic by a same proof used in corollary 3.5, implying the choice of path is redundant by the same corollary.

(b) Note that there exists a curve σ_1 like the solid one drawn below that misses all the points p_j as there are only finitely many of these points.



As before we define $F(z) := \int_{\sigma_1} f(z)dz$, with $\partial F/\partial y = if(z)$. Three applications of lemmas 3.9 or 3.10 then tells us that $\int_{\sigma_1} f(z)dz = \int_{\sigma_2} f(z)dz$, so that $\partial F/\partial x = f(z)$. The result follows. ☐

The integral over any closed curve is not always zero, as we will immediately see in the next section.

Definition 4.1. Let $\gamma : (a, b) \longrightarrow \mathbb{C}$ be a piecewise differentiable closed curve that does not pass through a point z_0 Then the *winding number* of γ about z_0 is defined to be

$$\eta(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}dz.$$

Proposition 4.2. *The value of $\eta(\gamma, z_0)$ is always an integer.*

Proof. Consider the function

$$h(t) := \int_a^t \frac{\gamma'(t)}{\gamma(t) - z_0}dt.$$

It is defined and continuous on $[a, b]$, with derivative

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

whenever $\gamma'(t)$ is continuous by the fundamental theorem of calculus. By the chain rule, this implies that the derivative of $g(t) := e^{-h(t)}(\gamma(t) - z_0)$ vanishes except at a finite set of points corresponding to the points that γ' is not continuous at. In any case, since $g(t)$ is continuous, it implies $g(t)$ is a constant. Hence

$$e^{-h(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}.$$

As $\gamma(a) = \gamma(b)$, this implies $e^{-h(\beta)} = 1$, so $h(\beta)$ is a multiple of 2π. Since $h(\beta) = 2\pi\eta(\gamma, z_0)$, we are done. □

The winding number $\eta(\gamma, z_0)$ relates to the topological notion as follow.

Proposition 4.3. *The following properties for $\eta(\gamma, z_0)$ holds.*

(a) $\eta(-\gamma, z_0) = -\eta(\gamma, z_0)$.
(b) *If γ lies inside of an open disk, then $\eta(\gamma, p) = 0$ for all points p in the complement of the open disk.*
(c) *As a function of z_0, the winding number $\eta(\gamma, z_0)$ is constant in each of the regions determined by γ, and is zero in the unbounded region.*
(d) *Let γ_1 and γ_2 be two homotopic paths such that there exists a homotopy between them that avoids z_0. Then $n(\gamma_1, z_0) = n(\gamma_2, z_0)$. Hence the winding number tells us how many times a curve winds about a chosen point geometrically.*

Proof. (a) This follows from part (d) of proposition 3.2.

(b) This is an immediate consequence of Cauchy's theorem for closed curves (part (a) of theorem 3.7).

(c) Let a and b be two points that lies in the same region determined by γ, and suppose that $\Re(a) \leq \Re(b)$. It suffices to assume that a and b can be connected by a single line segment l that does not intersect γ, since a and b can be connected by a polygonal segment in general. Notice the function $g(z) := (z-a)/(z-b)$ admits values in the nonpositive reals only at l, since $g(z) = N$ for $N \in \mathbb{R}$ if and only if $z = a/(1-N) - bN/(1-N)$, and the functions $1/(1-N)$ and $-N/(1-N)$ lies in the interval $[0, 1]$ precisely when $N \leq 0$.

Hence we can consider $\text{Log}\,((z-a)/(z-b))$, with derivative $\mathfrak{d} = (z-a)^{-1} - (z-b)^{-1}$. Since γ does not meet the line segment l, corollary 3.5 tells us that $\int_\gamma \mathfrak{d}$ depends only on its endpoints. But γ is a closed curve, implying

$$\int_\gamma \frac{1}{z-a}dz = \int_\gamma \frac{1}{z-b}dz.$$

(d) We can assume that γ_1, γ_2 has domain $[0, 1]$ since line integrals are invariant under a change of parameter. Let $H : [0, 1]\times[0, 1] \longrightarrow \mathbb{C}\setminus\{z_0\}$ be a homotopy from γ_1 to γ_2 away from z_0, and set $H_t := H(t, -)$. Define a map $\varphi : [0, 1] \longrightarrow \mathbb{Z}$ by

$$\varphi(t) := \frac{1}{2\pi i}\int_{H_t} \frac{1}{z - z_0}dz.$$

By definition of H, this is a continuous map, and is well-defined as none of H_t has z_0 in its image. Since $\mathbb{Z}$ is discrete, this implies φ is a constant, so indeed $\eta(\gamma_1, 0) = \eta(\gamma_2, z_0)$.

To see the claim on geometrical intuition, by translating z_0 to 0 it suffices to compute the winding number for $\gamma_n : [0, 1] \longrightarrow \mathbb{C}$, defined by $\gamma_n(t) = e^{2\pi i n t}$ for some $n \geq 1$. To see this, recall $\mathbb{C} \setminus \{0\}$ is homotopic to the circle, so that $\pi_1(\mathbb{C}\setminus\{0\})\cong\mathbb{Z}$. Therefore every curve on $\mathbb{C}\setminus\{z_0\}$ is homotopic to γ_n for some chosen n. A computation tells us that

$$\eta(\gamma_n,0)=\frac{1}{2\pi i}\int_0^1\frac{2\pi ine^{2\pi int}}{e^{2\pi it/n}}dt=n,$$

so indeed the winding number tells us how many times a curve winds about a chosen point. □

We are now ready to prove Cauchy's integral formula, which is one of the most useful computational formula in complex analysis (the other one being the residue theorem, to be discussed in section 7). A generalization of this will be given in section 11.

Theorem 4.4 (Cauchy's integral formula)**.** *Suppose f is analytic in an open disk D. Let γ be a closed curve in D.*

(a) For any $z_0\in D$ not on γ,

$$f(z_0)=\frac{1}{2\pi i\eta(\gamma,z_0)}\int_\gamma\frac{f(z)}{z-z_0}dz.$$

In particular, if γ is a circle C containing z_0 and contained in D, then

$$f(z_0)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-z_0}dz.$$

(b) The above part still holds if we replace D by D', obtained from D by omitting a finite number of points p_j satisfying the condition $\lim_{z\to p_j}(z-p_j)f(z)=0$, and if z_0 is chosen such that it is not on γ nor equal to any of the p_j's.

Proof. (a) Consider the function

$$F(z):=\frac{f(z)-f(z_0)}{z-z_0},$$

which is analytic away from z_0. At z_0, it satisfies the condition that $\lim_{z\to z_0}(z-z_0)F(z)=0$. Hence part (b) of theorem 3.7 tells us that $\int_\gamma F(z)dz=0$, whence the formula after recalling the definition of $\eta(\gamma,z_0)$.

(b) The condition of this part is rigged such that we can use the above argument, but using the assumption in part (b) of theorem 3.7, to prove the same statement. □

Cauchy's integral formula is followed with a list of useful results. The first one tells us complex functions are either not differentiable or smooth, and extends Cauchy's integral formula.

Theorem 4.5. *If f is an analytic function defined on an open set D, then it is infinitely differentiable. In particular,*

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz$$

for any circle C centered at z_0 with the region it bounds contained in D. Furthermore, if $|f|$ is bounded by a positive real M, then $|f^{(n)}(z_0)|\le n!Mr^{-n}$, where r is the radius of C as defined above.

To prove this theorem, we use a general lemma.

Lemma 4.6. *If φ is continuous on an arc γ, then the function*

$$F_n(z):=\int_\gamma\frac{\varphi(\zeta)}{(\zeta-z)^n}d\zeta$$

is analytic in each of the regions determined by γ, and its derivative is $F_n'(z)=nF_{n+1}(z)$.

Proof. We first show

$$\lim_{z\to z_0}\int_\gamma\frac{\varphi(\zeta)}{(\zeta-z)^n(\zeta-z_0)}d\zeta=\int_\gamma\frac{\varphi(\zeta)}{(\zeta-z)^{n+1}}d\zeta \tag{1}$$

for any integer $n\ge 0$ and z_0 away from γ. Choose δ_1 such that z is not on $\operatorname{im}\gamma$ for every $|z-z_0|<\delta_1$. Then $|\zeta-z_0|\ge\delta_1$ for all $\zeta\in\operatorname{im}\gamma$. By restricting to the smaller disk $|z-z_0|<\delta_1/2$, we see by the triangle inequality that

$$|\zeta-z|\ge|\zeta-z_0|-|z-z_0|>\frac{\delta_1}{2}$$

for every $\zeta \in \operatorname{im}\gamma$. Let $\epsilon > 0$, and choose $\delta > 0$ such that $2\delta \int_\gamma |\varphi(\zeta)||d\zeta| < \delta_1^{n+2}\epsilon$. If $|z - z_0| < \delta$, then

$$\begin{aligned}\left|\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}d\zeta\right| &\leq |z - z_0|\left|\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}(\zeta - z_0)}d\zeta\right| \\ &\leq |z - z_0|\frac{2}{\delta_1^{n+2}}\int_\gamma |\varphi(\zeta)||d\zeta| \\ &< \epsilon,\end{aligned}$$

giving what we wanted. Next we prove $F_n'(z) = nF_{n+1}(z)$ by induction. Consider the quotient

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)(\zeta - z_0)}d\zeta.$$

Equation 1 tells us that we can take the limit under the integral in the right hand side, so that $F_1'(z) = F_2(z)$. Assume now that $F_{n-1}'(z) = (n-1)F_n(z)$. Then

$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = \frac{1}{z - z_0}\left(\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}d\zeta - \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n}d\zeta\right) + \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n(\zeta - z_0)}d\zeta$$

Let us take the limit as z tends to z_0 in the above equality. The first term on the right hand side is then

$$\begin{aligned}\frac{d}{dz}\bigg|_{z=z_0}\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}d\zeta &= (n-1)\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^n(\zeta - z_0)}d\zeta \\ &= (n-1)F_{n+1}(z),\end{aligned}$$

and the second term on the right hand side tends to $F_{n+1}(z)$ by equation 1. Since the left hand side tends to $F_n'(z)$ by definition of the derivative, we are done. ☐

Proof of theorem 4.5. Assuming the explicit formula of $f^{(n)}(z_0)$, the estimate for $|f^{(n)}(z_0)|$ in case $|f| \leq M$ is immediate since

$$\begin{aligned}\left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz\right| &\leq \frac{n!}{2\pi}\int_C \left|\frac{f(z)}{(z - z_0)^{n+1}}\right||dz| \\ &\leq \frac{n!}{2\pi}\frac{2\pi r M}{r^{n+1}}.\end{aligned}$$

Hence it suffices to prove the integral formula of $f^{(n)}(z_0)$ is true. But this follows by lemma 4.6. ☐

The second result gives a partial converse to part (a) of theorem 3.7.

Corollary 4.7 (Morera)**.** *If f is defined and continuous on an open set D, and if $\int_\gamma f = 0$ for all closed curves γ in D, then f is analytic in D.*

Proof. The hypothesis implies that $\int_\gamma f$ depends only on the endpoints. Hence corollary 3.5 implies f is the derivative of an analytic function. But theorem 4.5 tells us that f is analytic as well. ☐

The third result ensures there are no nontrivial bounded entire functions, and gives us a quick proof of the fundamental theorem of algebra.

Corollary 4.8 (Liouville)**.** *Let f be an entire function.*

(a) If f is bounded, then it is constant.
(b) If there exists positive real constants A and B such that $|f| \leq A + B|z|^k$ for some integer $k \geq 0$, then f is a polynomial of degree at most k.

Proof. (a) Let f be a bounded entire function. Then $|f| \leq M$ for some real number M, and f' exists on the entire plane as well. Theorem 4.5 then tells us that

$$|f'(z_0)| \leq \frac{M}{r}$$

for every integer z_0 and any $r > 0$. This implies $f' \equiv 0$, so f is constant.

(b) The base case $k = 0$ is done in part (a). For the inductive step, consider the entire function

$$g(z) := \begin{cases} \dfrac{f(z) - f(0)}{z} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ f'(0) & \text{if } z = 0. \end{cases}$$

Then the hypothesis on f gives us constants A and B such that $|g| \leq A + B|z|^{k-1}$. By induction g is a polynomial of degree at most $k - 1$. Hence f is a polynomial of degree at most k. □

Corollary 4.9 (Fundamental theorem of algebra). *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a complex non-constant polynomial with complex coefficients such that $a_n \neq 0$. Then P has a complex zero. Consequently, P can be factored into n linear terms.*

Proof. We can assume $a_n = 1$ by dividing the coefficients. If $P(z)$ had no zeros, then its quotient $Q(z) = 1/P(z)$ will be entire. By Liouville's theorem it suffices to show that $Q(z)$ is bounded. Since $\lim_{z\to\infty} Q(z) = 0$, we can choose $\delta > 0$ such that $|Q(z) - Q(0)| < 1$ if $|z| > \delta$. If $|z| \leq \delta$, then $Q(z)$ is bounded as well by the Heine-Borel theorem, since the image of a compact subset under a continuous map is compact. Thus $Q(z)$ is bounded on the complex plane, implying $P(z)$ is constant, a contradiction. □

Liouville's theorem is the first of four major regularity theorems we will discuss. The other three are the maximum modulus principle and Picard's little and great theorems (see sections 9).

We end this section with an application of Cauchy's integral theorem that gives us an idea of what analytic continuations are.

Corollary 4.10 (Schwarz reflection principle). *Suppose f is analytic in a connected open set D contained in either the upper or lower half plane, with the boundary of its closure $\mathrm{cl}(D)$ being a segment L on the real axis. Suppose $f(z)$ is continuous on $\mathrm{cl}(D)$, with $f(r)$ real for all r lying on L. Then f can be extended to an analytic function defined by*

$$g(z) := \begin{cases} f(z) & \text{if } z \in D \cup L', \\ \overline{f(z)} & \text{if } z \in D^*, \end{cases}$$

where $D^* := \{\overline{z} : z \in D\}$ is the mirror image of D about the real axis, and L' is the set of points on L having a neighborhood contained in $D \cup D^* \cup (\mathrm{cl}(D) \cap \mathrm{cl}(D^*))$.

Proof. Notice that g is analytic in D as it agrees with f. In fact, g is analytic in D^* as well since

$$\frac{g(z+h) - g(z)}{h} = \overline{\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}}.$$

It remains to show that f is analytic on the line L. The proof goes in exactly the same way as that of lemma 3.9, so we simply give a sketch here. For any closed curve γ intersecting L, draw two lines parallel and ϵ-close to L. Use this to approximate $\int_\gamma f$ by continuity of f, giving zero. Finish by Morera's theorem. □

5. The Taylor and Laurent series

In this section we introduce power series expansions in complex analytic form. The key to their proof lies in the following useful observation, which will also be the motivation for our definitions in the next section.

Proposition 5.1. *Suppose f is analytic in D', obtained by omitting a point z_0 from an open set D. Then there exists an analytic function in D agreeing with f in D' if and only if $\lim_{z\to z_0}(z - z_0)f(z) = 0$. Furthermore, this function is uniquely determined near z_0 as*

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta,$$

where C is a circle in Ω containing z_0.

Proof. (⇒) This is clear, since f is analytic on D'.

(⇐) Using part (b) of Cauchy's integral formula (theorem 4.4), we see that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

for any circle C about z_0, and for any $z \neq z_0$. But the right hand side is even analytic and defined at z_0 since C avoids z_0, so we are done. The uniqueness is clear by Cauchy's integral theorem. □

We are now ready to prove Taylor's theorem, which is the basis for the validity of Taylor series.

Theorem 5.2 (Taylor)**.** *Let f be analytic in an open set D, and let $z_0 \in D$. Then we can write*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + f_n(z)(z - z_0)^n$$

for any integer $n > 0$, such that f_n is analytic in D with the property that $f^{(n)}(z_0) = n!f_n(z_0)$. In fact,

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^n(\zeta - z)}d\zeta$$

for any circle C about z_0 with the region it bounds contained in D, and such that z is contained inside C.

Proof. If we apply the previous proposition to the function

$$F(z) := \frac{f(z) - f(z_0)}{z - z_0}$$

considered in the proof of Cauchy's integral formula (theorem 4.4), then there exists an analytic function f_1 which equals $F(z)$ for $z \neq z_0$, and equals $f'(z_0)$ for $z = z_0$. In general, by replacing F by f_i and f by f_{i-1}, we defined analytic functions f_n such that

$$f_n(z) = \begin{cases} \dfrac{f_{n-1}(z) - f_{n-1}(z_0)}{z - z_0} & \text{if } z \neq z_0, \\ f'_{n-1}(z_0) & \text{if } z = z_0. \end{cases}$$

This iteration also tells us that $f_{n-1}(z) = f_{n-1}(z_0) + (z - z_0)f_n(z)$. With this, we get

$$f(z) = f(z_0) + (z - z_0)f_1(z_0) + \cdots + (z - z_0)^{n-1}f_{n-1}(z_0) + (z - z_0)^n f_n(z).$$

Differentiating n times gives us $f^{(n)}(z_0) = n!f_n(z_0)$, which proves the Taylor polynomial expansion. Let us now prove the integral formula of f_n. The case $z = z_0$ follows from theorem 4.5, so it suffices to consider the case when $z \neq z_0$. By Cauchy's integral formula and the Taylor polynomial expansion the we have

$$\begin{aligned} f_n(z) &= \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\int_C \frac{1}{(\zeta - z_0)^n(\zeta - z)}\left(f(\zeta) - f(z_0) - \frac{f'(z_0)}{1!}(\zeta - z_0) - \cdots - \frac{f^{(n-1)}(z_0)}{(n-1)!}(\zeta - z_0)^{n-1}\right)d\zeta. \end{aligned}$$

Note that the basic properties of winding numbers (proposition 4.3) tell us that

$$\int_C \frac{1}{(\zeta - z)(\zeta - z_0)}d\zeta = \frac{1}{z - z_0}\int_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right)d\zeta = 0$$

if z is inside C. Therefore all derivatives of the function above vanishes inside C, and lemma 4.6 tells us that

$$\int_C \frac{1}{(\zeta - z)^k(\zeta - z_0)}d\zeta = 0$$

for each integer $k \geq 1$. This reduces our Taylor polynomial expansion of $f_n(z)$ to what we claimed. □

The expression $f_n(z)(z - z_0)^n$ in the above theorem is sometimes called the n^{th} remainder term. Using the above theorem, we get the families Taylor series expansion in the complex case, as below.

Corollary 5.3 (Taylor series)**.** *Let f be analytic in an open set D, and let $z_0 \in D$. Then*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots,$$

and this expansion is valid in the largest open disk centered at z_0 and contained in D.

Proof. We will make use of the integral formula of f_n in Taylor's theorem. Let r be the radius of the circle C, and let $M := \max\{|f(z)| : z \in C\}$. The latter value exists by the Heine-Borel theorem. We obtain the estimate

$$\begin{aligned}|f_{n+1}(z)(z-z_0)^{n+1}| &\le \frac{1}{2\pi}|z-z_0|^{n+1}\int_C \frac{|f(\zeta)|}{|\zeta-z_0|^{n+1}|\zeta-z|}|d\zeta| \\ &\le \frac{M|z-z_0|^{n+1}}{r^n\left(r-|z-z_0|\right)}.\end{aligned}$$

Thus the remainder term $f_{n+1}(z)(z-z_0)^{n+1}$ tends to zero as z tends to z_0 in every disk $|z-z_0| \le r'$ with $r' < r$. We are done after observing that C can be made as large as f is defined on. ☐

The Taylor expansion examples that is done in the real case still holds in the complex case, even for the radius of convergence, so we will not repeat them here.

Taylor's theorem also implies the following, commonly known as the identity principle. The connectedness assumption is important as it gives the useful implication that makes this principle deserve some attention.

Corollary 5.4 (Identity Principle)**.** *Suppose f is analytic on a connected open set D. Then the following are equivalent.*

(a) $f \equiv 0$ *on* D.
(b) *The set* $Z := \{z \in D : f(z) = 0\}$ *has a limit point in* D.
(c) *There exists* $z_0 \in D$ *such that* $f^{(n)}(z_0) = 0$ *for every positive integer* n.

Proof. That (a) implies (b) is trivial. To see (b) implies (c), let p be a limit point of Z contained in D, and suppose for contradiction that $f^{(k)}(p) \neq 0$ for some positive integer p. Then, by applying the Taylor series expansion centered at p, we see that $f(z) = (z-p)^k g(z)$ for some function g that is analytic near p with $g(p) \neq 0$. Hence there exists a neighborhood N about p such that g is nonzero in N, implying f is nonzero in $N \setminus \{p\}$. This contradicts the assumption that p is a limit point of Z.

Finally, we show (c) implies (a). Letting $\mathcal{A} := \{z \in D : f^{(n)}(z) = 0 \text{ for all positive integers } n\}$, by assumption $\mathcal{A}$ is nonempty. It suffices to show that $\mathcal{A}$ is both open and closed, for the connectedness assumption on D implies $\mathcal{A}$ necessarily equals D. Let $z_0 \in \mathcal{A}$. To see $\mathcal{A}$ is open, we do the Taylor series expansion of f near z_0 to get

$$\begin{aligned}f(z) &= f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots \\ &= f(z_0).\end{aligned}$$

Since D is open, this expansion is valid on an open ball B about z_0. Thus $f(p) = f(z_0)$ for all $p \in B$, implying $p \in \mathcal{A}$. To see $D \setminus \mathcal{A}$ is open, by definition any $q \in D \setminus \mathcal{A}$ has $f^{(k)}(q) \neq 0$ for some positive integer k, so $f^{(k)}$ is nonzero in a neighborhood of q since $f^{(k)}$ is analytic. ☐

The identity principle gives us a surprising amount of rigidity on complex analytic functions, because it tells us that if two differentiable functions agree at a nontrivial open set, then they must agree everywhere. Furthermore, it tells us that every nonpolynomial function we can think of must not diverge in every direction. The proof of this is beautiful and uses some corollaries of the main theorems we have discussed.

Proposition 5.5. *If f is entire and $\lim_{z\to\infty} f(z) = \infty$, then f is a polynomial.*

Proof. The assumption implies that $|f(z)| > 1$ as $|z| > N$ for some real number N. This implies every zero of f lies in the compact set N, hence there are only finitely many of them by the identity principal as compactness implies sequential compactness in a metric space. Letting $\alpha_1, \dots, \alpha_n$ be the zeros of f, possibly with multiplicity, consider the entire function

$$g(z) := \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{f(z)}.$$

For $|z| \le N$, this function is bounded by $A = \left(\prod_{i=1}^n (N + |\alpha_i|)\right) / \min\{|f(z)| : |z| \le N\}$. For $|z| > N$, this function is bounded by $N^n|z|^n$ since every α_i has norm at most N. Therefore

$$|g(z)| \le A + N^n|z|^n,$$

implying g is a polynomial by Liouville's theorem (corollary 4.8). But by construction g is nonzero everywhere, implying it must be constant by the fundamental theorem of algebra (corollary 4.9). This implies what is claimed after a rearrangement. □

To end this section we discuss a related power series expansion called the Laurent series, which holds for analytic functions defined on an annulus. It is of mild interest since they can be used to compute residues, which we will explain further in section 7. In the proof of Laurent expansion we will also make use of the independence of homotopies on an annulus, which can be proven directly, but we defer to a general version to be proven later in section 11.

Theorem 5.6 (Laurent series)**.** *Suppose f is analytic on the annulus $\mathscr{A} = \{z : R_1 < |z - z_0| < R_2\}$ centered at z_0. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n,$$

where A_n can be computed as

$$A_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

with r satisfying $R_1 < r < R_2$. Furthermore, this expansion is unique and valid in $\mathscr{A}$.

Proof. Let us fix $z \in \mathscr{A}$, and fix two circles C_1 and C_2 in $\mathscr{A}$ with radii r_1 and r_2 respectively, such that $R_1 < r_1 < |z - z_0| < r_2 < R_2$. Note that it suffices to consider the case $z_0 = 0$, since the general case follows by translation. Consider the function

$$F(w) = \frac{f(w) - f(z)}{w - z}.$$

We saw in the proof of theorem 4.4 that F is analytic in $\mathscr{A}$, so $\int_{C_2 - C_1} F(w) dw = 0$ by Cauchy's theorem (see example 11.5), so

$$\int_{C_2 - C_1} \frac{f(w)}{w - z} dw = \int_{C_2 - C_1} \frac{f(z)}{w - z} dw = 2\pi i f(z),$$

where we used an easy winding number computation in the final equality. Hence $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = -\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w - z} dw \qquad \text{and} \qquad f_2(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w - z} dw.$$

On C_2, we see that $|w| > |z|$, and on C_1, we see that $|w| < |z|$. Hence the Taylor series expansion tells us that

$$\frac{1}{w - z} = \begin{cases} \dfrac{1}{w(1 - z/w)} = \dfrac{1}{w}\left(1 + \dfrac{z}{w} + \cdots + \dfrac{z^k}{w^k} + \cdots\right) & \text{if } z \text{ is on } C_2, \\ -\dfrac{1}{z(1 - w/z)} = -\dfrac{1}{z}\left(1 + \dfrac{w}{z} + \cdots + \dfrac{w^k}{z^k} + \cdots\right) & \text{if } z \text{ is on } C_1. \end{cases}$$

Furthermore, each of the two power series expansions are clearly uniformly convergent. Hence part (g) of proposition 3.2 tells us that

$$f_1(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \left(\int_{C_1} \frac{f(w)}{w^{k+1}} dw\right) z^k \qquad \text{and} \qquad f_2(z) = \frac{1}{2\pi i} \sum_{k=1}^{\infty} \left(\int_{C_2} f(w) w^k dw\right) \frac{1}{z^{k+1}},$$

giving the Laurent expansion. Notice that when computing A_n we made used of C_1 and C_2 on different occasions. However, by independence of homotopies (example 11.5) we can in fact just choose any circle C centered at z_0 and lying in $\mathcal{A}$.

We now show uniqueness of Laurent expansions. Again we assume $z_0 = 0$. Consider any Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Since it is necessarily uniformly convergent along γ, part (g) of proposition 3.2 again tells us that

$$\int_C \frac{f(z)}{z^{k+1}} dz = \sum_{n=-\infty}^{\infty} \int_C a_n z^{n-k-1} dz,$$

where C is a circle of radius R as above. Writing $C = \{Re^{i\theta} : 0 \le \theta \le 2\pi\}$, we see that

$$\int_C z^{n-k-1} dz = \begin{cases} \int_C dz = 2\pi i & \text{if } n = k+1, \\ \left. \dfrac{(iRe^{i\theta})(R^{n-k-1}e^{i(n-k-1)\theta})}{i(n-k-1)} \right|_{\theta=0}^{2\pi} = 0 & \text{if } n \neq k+1. \end{cases}$$

Hence $a_n = A_n$, proving uniqueness. □

One should note that Taylor series and Laurent series expansions are generally not the same. The former gives an expansion that is valid locally, while the latter attempts to achieve an expansion that is valid on a large region without analyzing the local behavior.

Example 5.7. A function may have more than one Laurent expansion, but they are necessarily defined on different annuli by the uniqueness claim in the previous theorem. For example, consider the rational function

$$f(z) = \frac{1}{z(1+z^2)}.$$

We do not look for Laurent expansions along the circles $|z| = 0$ and $|z| = 1$ as f has zeros over there. As for the other cases, we use the usual expansion of $(1-x)^{-1}$ that is valid on $|x| < 1$ to see that the two annuli we are loooking for are $0 < |z| < 1$ and $|z| > 1$, with Laurent expansions

$$f(z) = \begin{cases} \dfrac{1}{z}\left(1 - z^2 + \cdots + \dfrac{(-1)^n}{z^{2n}} + \cdots\right) & \text{if } 0 < |z| < 1, \\ \dfrac{1}{z^3}\left(1 - \dfrac{1}{z^2} + \cdots + \dfrac{(-1)^n}{z^{2n+3}} + \cdots\right) & \text{if } |z| > 1. \end{cases}$$

This gives two different Laurent expansions on different domains, as we would expect.

6. Singularities, zeros, and poles

Definition 6.1. If z_0 is a zero of f, its *order* is the positive integer n for which we can expand f as a Taylor series about z_0 and write $f(z) - f(z_0) = (z-a)^n g_n(z)$ such that $g_n(a) \neq 0$.

Definition 6.2. Suppose f is a function analytic in a neighborhood U of z_0, except at z_0 itself. Then z_0 is called an *isolated singularity* of f. For an isolated singularity, we classify it into three types as follow.

- Say z_0 is a *removable singularity* of f if there exists an analytic function defined on U such that $f(z) = g(z)$ for all $z \in U \setminus \{z_0\}$. By a slight abuse of notation, we call the value of the extended function at z_0 to be $f(a)$.
- Say z_0 is a *pole* of f if $\lim_{z\to z_0} f(z) = \infty$. By proposition 5.1, the nonzero function $g(z) = 1/f(z)$ has a removable singularity at z_0, so we can expand f as a Taylor series about z_0 and write $g(z) = (z-z_0)^n g_n(z)$ such $g_n(z_0) \neq 0$. This positive integer n is called the *order* of the pole z_0. If the order of a pole z_0 equals one, then it is called a *simple pole*.
- Say z_0 is an *essential singularity* of f if it is neither a removable singularity nor a pole of f.

The undefined point $f(\infty)$ would be treated as an isolated singularity, and its characterization is determined by looking at the singularity of $g(z) = f(1/z)$ at $z = 0$.

Definition 6.3. A function that is analytic in an open set except for poles is said to be *meromorphic*.

We can detect the characterization of each isolated singularity using the proposition below.

Proposition 6.4. *Let z_0 be a point on the region an analytic function f is defined on.*

(a) *z_0 is a removable singularity if* $\lim_{z\to z_0}(z-z_0)f(z) = 0$.

(b) *Suppose there exists a positive integer n such that*

$$\lim_{z\to z_0} |(z-z_0)|^{\alpha}|f(z)| = \begin{cases} \infty & \text{if } \alpha < n, \\ 0 & \text{if } \alpha > n, \\ \text{finite} & \text{if } \alpha = n. \end{cases}$$

If $n = 0$, then f is analytic but nonzero at z_0. If n is negative, then z_0 is a zero of f. If n is positive, then z_0 is a pole of f. This positive integer n gives the order of z_0 in case z_0 is a zero or a pole.

(c) z_0 *is an essential singularity if* $\lim_{z\to z_0} |(z-z_0)|^\alpha |f(z)|$ *is finite and nonzero for every real number* α.

Proof. (a) This is the content of proposition 5.1.

(b) The claim for $n=0$ is clear, so we assume $n \neq 0$. If $n<0$, we expand f about z_0 (corollary 5.3) to get $f(z) = (z-z_0)^n g(z)$, where $g(z_0) \neq 0$. If $n>0$, we expand f about z_0 to get $(z-z_0)^n f(z) = g(z)$, where $g(z_0) \neq 0$. The respective claims are now clear.

(c) This follows directly from the previous part. □

Essential singularities are in general complicated to understand, and the following theorem tells us why.

Theorem 6.5 (Casorati-Weierstrass)**.** *Suppose* z_0 *is an essential singularity of an analytic function* f*, and let* U *be a neighborhood of* z_0*. Then the set* $f(U \setminus \{z_0\})$ *is dense in the complex plane.*

Proof. If not, there exists a complex number c such that $|f(z)-c| > \delta$ for some $\delta > 0$, and for all $z \in U \setminus z_0$. Hence $\lim_{z\to z_0} |z-z_0|^{-1}|f(z)-c| = \infty$, and z_0 is not an essential singularity of f. Hence proposition 6.4 tells us that $\lim_{z\to z_0} |z-z_0|^\alpha |f(z)-c| = 0$ for some real number α. As $\lim_{z\to z_0} |z-z_0|^\alpha c = 0$, the triangle inequality implies $\lim_{z\to z_0} |z-z_0|^\alpha |f(z)| = 0$, so z_0 is not an essential singularity of f, a contradiction. □

The Casorati-Weierstrass has a much stronger form, known as the great Picard theorem, which we will discuss in section 9. For now, we move on to generalizing the two Cauchy theorems.

7. Residues

The residue theorem is an extremely useful computational formula, the other one being Cauchy's integral formula generalized in the previous section.

Definition 7.1. The residue of f at an isolated singularity z_0 is defined to be

$$\operatorname{Res}_{z_0} f(z) := \frac{1}{2\pi i}\int_C f,$$

where C is a small circle centered at z_0 and containing no other singularities in the region it bounds.

There is no general way to compute residues for essential singularities. However, there are two useful ways to compute residues in case our point is not an essential singularity.

Proposition 7.2. *Let* z_0 *be an isolated singularity of* f*.*

(a) The residue of a zero at f *equals zero.*
(b) If we consider the Laurent expansion $f = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n$ *in a deleted neighborhood of* z_0*, then* $\operatorname{Res}_{z_0} f(z) = A_{-1}$.
(c) If z_0 *is a pole of order* k *for* f*, then*

$$\operatorname{Res}_{z_0} f(z) = \frac{1}{(k-1)!}\lim_{z\to z_0}\left(\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)\right).$$

In particular, if z_0 *is a simple pole, then* $\operatorname{Res}_{z_0} f(z) = (z-z_0)f(z_0)$.

Proof. (a) This is clear from Cauchy's theorem on closed curves.

(b) This is immediate from theorem 5.6 by considering $R_1 = 0$.

(c) The Laurent expansion about z_0 in this case is $\sum_{n=-k}^{\infty} A_n (z-z_0)^n$. Therefore the coefficient of $(z-z_0)^{k-1}$ in $(z-z_0)^k f(z)$ is A_{-1}. Hence the result follows from part (b) after differentiation. □

With the definition of a residue set up, we are now ready to state the residue theorem. The proof is almost tautological, and its importance lies in its applications.

Definition 7.3. A cycle γ in an open set D is *homologous to zero* with respect to D if the winding number $\eta(\gamma, z_0) = 0$ for all z_0 in the complement of D.

Theorem 7.4 (Residue theorem)**.** *Let* f *be analytic in an open set* D *except for isolated singularities* z_j*. Then*

$$\frac{1}{2\pi i}\int_\gamma f = \sum_i \eta(\gamma, z_j)\operatorname{Res}_{z_j} f(z)$$

for any cycle γ *which is homologous to zero in* D *and avoid each* z_j*.*

Proof. We first assume there are only finitely many isolated singularities z_j. Note that γ homologous to zero in D implies that $\sum_i \eta(\gamma, z_j) C_j$ is homologous to zero in D as well, where C_j is any small circle centered at z_j and containing no other singularities in the region it bounds. Hence, by definition,

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\sum_j \eta(\gamma, z_j)\int_{C_j} f = \sum_i \eta(\gamma, z_j)\operatorname{Res}_{z_j} f(z).$$

In the general case where there are possibly infinitely many isolated singularities, we need to show that there are only finitely many of them with nonzero winding number with respect to γ. This is because the set of all points p with $\eta(\gamma, p) = 0$ is open and contains all points outside of a large enough circle. Hence its complement is a closed and bounded set, implying compactness by the Heine-Borel theorem. Since every point q has a neighborhood containing no other isolated singularities by definition, this implies $\eta(\gamma, q) \neq 0$ for finitely many such q's, as desired. □

Residues calculus is an art. We compute two examples – see books like [4] for a lot more examples.

Example 7.5. Consider the meromorphic function

$$f(z) := \frac{e^z}{(z-a)(z-b)},$$

which has simple poles at a and b if $a \neq b$, and a pole of order two at a if $a = b$. In the former case we see that $\operatorname{Res}_a f(z) = e^a/(a-b)$ and $\operatorname{Res}_b f(z) = e^b/(b-a)$. In the latter case we use part (c) of proposition 7.2 to see that

$$Res_{z=a} f(z) = \frac{1}{(2-1)!}\lim_{z\to a}\left(\frac{d}{dz}\left((z-a)^2\frac{e^z}{(z-a)^2}\right)\right) = e^a.$$

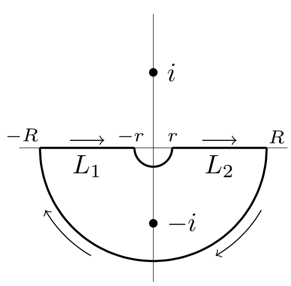
Hence, if we integrate along a circle C that encloses both a and b, then

$$\int_C f = \begin{cases} 2\pi i\left(\dfrac{e^a - e^b}{a-b}\right) & \text{if } a \neq b, \\ 2\pi i e^a & \text{if } a = b. \end{cases}$$

Example 7.6. We can use the residue theorem to help us compute real integrals. For example, let us show that

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi}{\sqrt{2}}.$$

In order to compute this integral, consider closed curve C below.



Let the inner and outer semicircle of radii r and R on the curve be C_1 and C_2 respectively. Observe that

$$\begin{aligned}\int_{L_1}\frac{\sqrt{z}}{1+z^2}dz + \int_{L_2}\frac{\sqrt{z}}{1+z^2}dz &= \int_{-R}^{-r}\frac{\sqrt{l}}{1+l^2}dl + \int_r^R \frac{\sqrt{l}}{1+l^2}dl \\ &= -\int_R^r \frac{e^{\frac{1}{2}\log(-l)}}{1+l^2}dl + \int_r^R\frac{\sqrt{l}}{1+l^2}dl \\ &= \int_r^R \frac{e^{-\pi i/2}\sqrt{l}}{1+l^2}dl + \int_r^R \frac{\sqrt{l}}{1+l^2}dl \\ &= \left(1+e^{-\pi i/2}\right)\int_{L_2}\frac{\sqrt{z}}{1+z^2}dz,\end{aligned}$$

so this integral only depends on the length of L_2. Also observe that

$$\left|\int_{C_1} \frac{\sqrt{z}}{1+z^2} dz\right| \leq \frac{2\pi r\sqrt{r}}{1+r^2} \qquad \text{and} \qquad \left|\int_{C_2} \frac{\sqrt{z}}{1+z^2} dz\right| \leq \frac{2\pi R\sqrt{R}}{1+R^2}.$$

Hence, as r tends to 0 and R tends to ∞, we see the line integrals along C_1 and C_2 tends to zero. On the other hand, after noting $\eta(C,-i) = -1$, the residue theorem tells us that

$$\frac{1}{2\pi i}\int_C \frac{\sqrt{z}}{1+z^2} dz = -\lim_{z\to -i}(z+i)\frac{\sqrt{z}}{1+z^2} = \frac{e^{-\pi/4i}}{2i},$$

which is independent of r and R. Hence, letting $r \to 0$ and $R \to \infty$, we have

$$\left(1+e^{-\pi i/2}\right)\int_0^\infty \frac{\sqrt{x}}{1+x^2} dz = \frac{2\pi i e^{-\pi/4i}}{2i},$$

giving us the desired computation after rearranging.

As we can see, much care must be taken when computing integrals using the residue theorem. In particular, we always specify a branch, construct a closed curve that lies entirely in the branch, and compute the winding number of the closed curve we integrate on (which will usually be either 1 or -1).

Let us now give another application of the residue theorem to count zeros and poles.

Theorem 7.7 (Argument principle)**.** *Suppose f is meromorphic in an open set D with zeros a_j and poles b_k of orders $\mathfrak{z}_j$ and $\mathfrak{p}_k$ respectively. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_j \mathfrak{z}_j \eta(\gamma, a_j) - \sum_k \mathfrak{p}_k \eta(\gamma, b_k)$$

for every cycle γ which is homologous to zero in D and avoids each of the zeros and poles.

Proof. For each a_j, let us write its Taylor series expansion there as $f(z) = (z-a)^{\mathfrak{z}_j} g_{a_j}(z)$ with $g_{a_j}(a_j) \neq 0$. A computation tells us that $f'(z) = \mathfrak{z}_j(z-a)^{\mathfrak{z}_j - 1} g_{a_j}(z) + (z-a)^{\mathfrak{z}_j} g'_{a_j}(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{\mathfrak{z}_j}{z-a_j} + \frac{g'_{a_j}(z)}{g_{a_j}(z)}.$$

Since the second term on the right hand side is analytic near a_j, we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \frac{1}{2\pi i}\int_\gamma \frac{\mathfrak{z}_j}{z-a_j} dz = \mathfrak{z}_j \eta(\gamma, a_j).$$

Doing the same Taylor series expansion with the poles, we have $f(z) = (z-a)^{-\mathfrak{p}_k} g_{b_k}(z)$ with $g_{b_k}(b_k) \neq 0$ and $f'(z) = -\mathfrak{p}_j(z-a)^{-\mathfrak{p}_j - 1} g_{a_j}(z) + (z-a)^{\mathfrak{p}_j} g'_{a_j}(z)$. Thus we have a same formula as above, except with a negative sign due to our expansion. The theorem follows from the residue theorem (theorem 7.4). $\square$

The argument principle allows us to give three theoretical applications. The first application tells us we can define single-valued branches of the logarithm and n^{th}-zero functions given a sufficiently nice open set.

Corollary 7.8 (Existence of single-valued branches)**.** *Let f be analytic and nonzero in a connected and simply-connected open set D. Then we can define single-valued analytic branches of* $\log f$ *and* $\sqrt[n]{f}$ *on D.*

Proof. Indeed, the argument principle implies $\int_\gamma f'/f$ is zero for every closed curve in D (recall closed curves in a connected and simply-connected open set are always homologous to zero). Hence corollary 3.5 gives us the existence of an analytic function F in D with $F' = f'/f$. Considering $g(z) := f(z)e^{-F(z)}$, we see that

$$g'(z) = f'(z)e^{-F(z)} - f(z)F'(z)e^{-F(z)} = 0,$$

so in fact g is a constant. Thus we can choose any point $z_0 \in D$ and any value $\log f(z_0)$, and we will get

$$f(z) = f(z_0)e^{-F(z_0)}e^{F(z)} = e^{F(z)-F(z_0)+\log f(z_0)}.$$

Hence we can define $\log f$ and $\sqrt[n]{f} = \exp\left(\frac{1}{n}\log f\right)$ on D as

$$\log f(z) = F(z) - F(z_0) + \log f(z_0) \qquad \text{and} \qquad \sqrt[n]{f(z)} = \exp\left(\frac{F(z)-F(z_0)+\log f(z_0)}{n}\right),$$

which are certainly single-valued branches on D. $\square$

The second application tells us how zeros of analytic functions behave locally.

Corollary 7.9. *Let f be analytic with an isolated zero at z_0. Let $C : I \longrightarrow \mathbb{C}$ be a counterclockwise circle about z_0 with the open disk D it bounds contained in the domain of definition of f, and such that z_0 is the only zero of f in C of order n.*

(a) For any $p, q \in D$, we have that $\eta(f \circ C, f(p)) = \eta(f \circ C, f(q))$. Consequently, the equations $f(z) = f(p)$ and $f(z) = f(q)$ both has n zeros, counted with multiplicity.
(b) There exists a neighborhood N of $f(z_0)$ such that the equation $f(z) = p$ has exactly n zeros in D for all $p \in N \setminus \{f(z_0)\}$.

Proof. (a) Since D is connected, its image under f is still connected. Hence $f(p)$ and $f(q)$ lies in the same connected component determined by $f \circ C$, so $\eta(f \circ C, f(p)) = \eta(f \circ C, f(q))$ by part (c) of proposition 4.3. The second assertion follows by the argument principle, since

$$\begin{aligned}
\int_C \frac{f'(z)}{f(z) - f(p)} dz &= \int_0^1 \frac{f'(C(t))C'(t)}{(f \circ C)(t) - f(p)} dt \\
&= \int_{f \circ C} \frac{1}{z - f(p)} dz \\
&= \eta(f \circ C, f(p)) \\
&= \eta(f \circ C, f(q)) \\
&= \int_{f \circ C} \frac{1}{z - f(q)} dz \\
&= \int_0^1 \frac{f'(C(t))C'(t)}{(f \circ C)(t) - f(q)} dt \\
&= \int_C \frac{f'(z)}{f(z) - f(q)} dz.
\end{aligned}$$

(b) Part (a) asserts $f(z) = p$ has n zeros counted with multiplicity. We need to show each of the n zeros are distinct. To do this, by Taylor expansion (corollary 5.3) it suffices to find a disk D' about z_0 such that $f'(p) \neq 0$ for all $p \in D' \setminus \{z_0\}$. As f is nonconstant (else z_0 is not an isolated zero), such a disk D' exists by the identity principle (corollary 5.4). $\square$

The third application gives another nice property of complex functions, that if it is injective then its derivative vanishes nowhere. This is certainly not true in the real case, an easy example being the cubic curve $y = x^3$. This fact will be important later when we study conformal mappings in section 10.

Corollary 7.10 (Nonvanishing derivatives of analytic functions)**.** *Let f be a injective analytic function on an open set D.*

(a) For every $z_0 \in D$, necessarily $f'(z_0) \neq 0$. Conversely, if f' vanishes nowhere, then for every $z_0 \in D$ there exists a neighborhood U_{z_0} of z_0 such that z_0 is the only zero of the equation $f(z) = f(z_0)$.
(b) The inverse $f^{-1} : f(D) \longrightarrow D$ of f has derivative $(f^{-1})'(z) = 1/f'(f^{-1}(z))$ for each $z \in f(D)$.

Proof. (a) Suppose $f'(z_0) = 0$ for some $z_0 \in D$, and consider $g(z) := f(z) - f(z_0)$. Note that the zeros of g are discrete by the identity principle (corollary 5.4) since g is injective, hence nonconstant. Thus we can assume z_0 is the only zero of g by restricting to an appropriate domain $D' \subset D$. By considering the Taylor expansion of f near z_0, we see that z_0 is a zero of g of order $k > 1$. Part (b) of the previous corollary then tells us that there exists a point $p \in f(D)$ such that $f(z) = p$ has $k > 1$ distinct zeros. This implies f is not injective.

Conversely, suppose that for every $z_0 \in D$ and every neighborhood U_{z_0} of z_0 there exists a zero of $f(z) = f(z_0)$ other than z_0. Then we can find a sequence of complex numbers in D that converges to z_0 and such that their value under f equals $f(z_0)$. This implies f is constant by the identity principle (corollary 5.4), contradicting the assumption that f' vanishes nowhere.

(b) Simply do the chain rule to get $1 = \mathrm{id}'_{f(D)}(z) = (f \circ f^{-1})'(z) = f'(f^{-1}(z))(f^{-1})'(z)$. As f' vanishes nowhere by the part (a), we can divide by $f'(f^{-1}(z))$ throughout to get what we want. ☐

To conclude this section, we give an application of the argument principle on comparing the zeros of analytic functions.

Corollary 7.11 (Rouche's theorem)**.** *Let γ be homologous to zero in an open set D and such that*

$$\eta(\gamma, z) = \begin{cases} 1 & \text{if } z \text{ is a zero of } f \text{ enclosed by } \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose f and g are both analytic in D and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then f and g have the same number of zeros enclosed by γ, counted with multiplicity.

Proof. Notice the assumption implies f and g have no poles in D, and furthermore the inequality assumption implies f and g must not have any zeros on γ. So we can write $g = (g/f)f$, implying

$$\int_\gamma \frac{g'}{g} = \int_\gamma \frac{(g/f)'f + (g/f)f'}{g} = \int_\gamma \frac{(g/f)'}{g/f} + \int_\gamma \frac{f'}{f}.$$

If we can show the first integral on the right hand side is zero, then we are done by the argument principle (theorem 7.7). By the inequality assumption g/f also satisfies the inequality

$$\left| 1 - \frac{g(z)}{f(z)} \right| < 1$$

on γ, implying the values of g/f on γ are contained in the open unit disk centered at 1. Let $\gamma = \gamma_1 + \cdots + \gamma_n$, where some of the γ_i's may be equal to each other, and let $\gamma_i : [\alpha_i, \beta_i] \longrightarrow \mathbb{C}$ be the curve. Then a computation tells us that

$$\begin{aligned} \int_\gamma \frac{(g/f)'}{g/f} &= \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} \frac{((g/f)' \circ \gamma_i)(t)\gamma_i'(t)}{((g/f) \circ \gamma_i)(t)} dt \\ &= \sum_{i=1}^n \int_{(g/f)\circ\gamma_i} \frac{1}{z} dz \\ &= \sum_{i=1}^n \eta((g/f) \circ \gamma_i, z) \\ &= 0, \end{aligned}$$

where the last equality is because $(g/f) \circ \gamma_i$ is contained in the open unit disk centered at 1, hence having the origin in the unbounded region (see part (c) of proposition 4.3). We are done. ☐

The most common test using Rouche's theorem is to find the number of zeros of a polynomial inside a closed disk, which we demonstrate using an example.

Example 7.12 (Zeros of polynomials)**.** Consider the complex polynomial $z^{10} - z^7 + 4z^2 - 1$. Let us show that it has exactly two zeros inside the open unit disk centered at the origin. To see this, we use the unit circle as our closed curve, and let $f = 4z^2$ and $g = z^{10} - z^7 + 4z^2 - 1$. Then $f - g = z^{10} - z^7 - 1$, and we aim to show that $|f - g| < |f| = 4$. But

$$|z^{10} - z^7 - 1| \leq |z|^{10} + |z|^7 + 1 = 3,$$

which is certainly less than 4. Hence $z^{10} - z^7 + 4z^2 - 1$ has the same number of zeros as $4z^2$ in the open unit disk by Rouche's theorem counted up to multiplicity, which is two. Alternatively, we could have just used the triangle inequality to conclude.

In fact, by using the same f and g, we can show $z^{10} - z^7 + 4z^2 - 1$ has two zeros inside the open disk centered at the origin of radius $\sqrt[8]{4/3}$. In particular, it has two zeros on the closed unit disk centered at the origin. To see this, we consider $z = re^{i\theta}$ for $r \geq 1$. Then, in order to fulfill the assumptions of Rouche's theorem, it suffices to have $3r^{10} < 4r^2$, which gives us what we want.

8. The maximum modulus principle

In this section we will discuss our section regularity theorem in complex analysis, called the maximum modulus principle. Like the section on Cauchy's integral formula (section 4), this is followed by a list of useful corollaries.

Theorem 8.1 (Maximum modulus principle)**.** *If f is analytic and nonconstant in a connected open set D, then its absolute value $|f|$ has no local maximum in D, i.e. for each $z_0 \in D$, there is no neighborhood N of z_0 such that $|f(z_0)| \geq |f(z)|$ for all $z \in N$.*

Proof. Fix $z_0 \in D$, and fix a neighborhood N of z_0. Let C be a circle in D centered at z_0 and contained in H, and let r be the radius of C. Parametrizing C by $\{z_0 + re^{i\theta} : 0 \leq \theta \leq 2\pi\}$, Cauchy's integral formula (theorem 4.4) tells us that

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) ire^{i\theta}}{(z_0 + re^{i\theta}) - z_0} d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \end{aligned}$$

and we see that r can be chosen independently. Therefore $|f(z_0)| \leq \max\{|f(z + re^{i\theta}| : 0 \leq \theta \leq 2\pi\}$ for each r. We aim to show at least one of these is a strict inequality. If all of these were an equality, then the integral above tells us that $|f|$ would be constant on each circle C centered at z_0 and contained in N. This implies f is constant throughout D by the identity principle (corollary 5.4). We get a contradiction as f is nonconstant by assumption. □

We now list three corollaries of the maximum modulus principle.

Corollary 8.2 (Minimum modulus principle)**.** *If f is analytic and nonconstant in a connected open set D, then no points of D is a relative minimum of f unless $f(z) = 0$.*

Proof. This follows directly from the maximum modulus principle by considering $1/f$ locally about a point in D that is not a zero of f. □

Corollary 8.3 (Open mapping theorem)**.** *Let D be a connected open set. A nonconstant analytic function $f : D \longrightarrow \mathbb{C}$ maps open sets onto open sets.*

Proof. We need to show that if f is nonconstant and analytic at z_0, then there exists an open disk D containing z_0 such that $f(D)$ contains an open disk centered at z_0. By translation it suffices to let $f(z_0) = 0$. The identity principle (corollary 5.4) tells us that there exists a circle C about z_0 with $f(z) \neq 0$ for every $z \in C$, else we will be able to find a sequence converging to z_0 with images under f identically zero. Letting D be the open disk that C bounds, we will show that $f(D)$ contains the ball $B_\epsilon(f(z_0))$ about $f(z_0)$, where $2\epsilon = \min\{|f(z)| : z \in C\}$.

Let $w \in B_\epsilon(f(z_0))$. Then the triangle inequality tells us that

$$|f(z) - w| \geq |f(z)| - |w| \geq \epsilon$$

for each $z \in C$, and at $f(z_0)$ we have

$$|f(z_0) - w| = |-w| < \epsilon.$$

This implies $|f(z) - w|$ is in $f(D) \setminus f(C)$. If we consider the map $g : \mathrm{cl}(D) \longrightarrow \mathbb{R}$ by $g(z) = |f(z) - w|$, then it is continuous by the continuity of f and the triangle inequality. Since $\mathrm{cl}(D)$ is compact, $\mathrm{im}\, g$ is closed and bounded by the Heine-Borel theorem, so the least upper bound property ensures $\mathrm{im}\, g$ has a minimum (this is also called the *extreme value theorem*). Our argument above tells us that the minimum is achieved away from $f(C)$, so the minimum modulus principle tells us that $|f(m) - w| = 0$ for some $m \in D$. This implies $w = f(m)$, so $B_\epsilon(f(z_0)) \subset f(D)$, as desired. □

Corollary 8.4 (Schwarz's lemma)**.** *Suppose f satisfies the following three conditions:*

- *f is analytic on the open disk $D = \{z : |z| < 1\}$,*
- *$|f(z)| \leq 1$,*
- *$f(0) = 0$.*

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality holds in either of these two inequalities for some $z_0 \in D$ if and only if $f(z) = e^{i\theta} z$ for some real constant θ.

Proof. Consider the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } 0 < |z| < 1, \\ f'(0) & \text{if } z = 0. \end{cases}$$

This function is analytic by proposition 5.1, and $|g(z)| = |f(z)|/|z| < 1/r$ on the circle of radius r. Letting r tend to 1 and applying the maximum modulus principle (theorem 8.1) on D, we see that $|g| \leq 1$ on $\mathrm{cl}(D)$. This proves the desired inequalities. Now suppose $|g(z_0)| = 1$ for some $z_0 \in D$. By the maximum modulus principle again g is constant of modulus 1, implying $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. The converse of this claim is clear. □

Schwarz's lemma, together with Schwarz's reflection principle (corollary 4.10), will be used when we study important classes of conformal maps in section 10.

In this section we study Phragmén-Lindelöf theorem, which generalizes maximum modulus principle. For motivation, let us note that maximum modulus principle implies the following.

Corollary 8.5. *Let f be continuous on an open set D.*

(a) If f is analytic on a bounded connected open subset D' of D with $\mathrm{cl}(D') \subset D$, then the maximum modulus of f lies on the boundary $\partial D'$.

(b) Suppose D is connected. If there are two real constants M_1 and M_2 such that $|f| \leq M_1$ for all $z \in \partial D$ and $|f| \leq M_2$ for all $z \in D$, then in fact $|f| \leq M_1$ for all $z \in D$.

Proof. (a) Since $\mathrm{cl}\, D'$ is compact and bounded, the image set $f(\mathrm{cl}(D'))$ is closed and bounded as well. Hence $f(\mathrm{cl}(D'))$ has a maximum modulus by the extreme value theorem (see the proof of corollary 8.3). By the maximum modulus principle, this maximum modulus must lie on the boundary $\partial D'$ if f is nonconstant on D'. If $|f|$ is constant on $D'$with value c, then continuity on D ensures $f(d') = c$ for any $d' \in \partial D'$, so the conclusion still holds.

(b) We may assume f is not constant, else the result is clearly true. Thus, after diving f by a constant, we can let $M_1 = 1$. We may also assume D is not bounded, since the other case follows from the next part of this corollary. Fixing some $a \in D$, we will once again consider the function

$$g(z) := \frac{f(z) - f(a)}{z - a}.$$

After defining $g(a)$ using proposition 5.1, this function is analytic throughout D. Since $|f|$ is bounded in the unbounded D, we see that $g \to 0$ as $z \to \infty$. Hence, by the same argument as the one used in proving the fundamental theorem of algebra (corollary 4.9), we see that $|g| \leq K$ for some real constant K.

Let us set $h = f^N g$ for any positive integer N, and let $D_R := \{z \in D : |z| \leq R\}$. Since g tends to 0 as z tends to infinity, we are able to choose R_0 large enough such that $|h| \leq K$ along ∂D_R for each $R \geq R_0$. By varying R, we can use corollary 8.5 to conclude $|h| \leq K$ for every point in D as well. For each $z_0 \in D$ that is not a zero of g, we can write

$$|f(z_0)| \leq \left| \frac{K}{g(z_0)} \right|^{1/N}.$$

Letting N tend to infinity, we see $|f(z_0)| \leq 1$. As for the zeros of g, note that, since we assumed f is nonconstant, the identity principle (corollary 5.4) assures us that they must be discrete. Hence continuity tells us that $|f| \leq 1$ at every zero as well, so $|f| \leq 1$ throughout D. □

The above corollary does not hold if we do not assume D' were unbounded, or if we do not assume any boundedness conditions in the interior of D. For example, the exponential function e^z is analytic on the entire plane, therefore on the right half plane $H_r := \{z \in \mathbb{C} : \Re(z) > 0\}$. It admits the constant modulus 1 on the boundary of H_r, but this is certainly not the maximum modulus on $\mathrm{cl}\, H_r$. However, if we give ourselves some boundedness conditions, we are able to give stronger maximum-modulus-like results. Before that, it is convenient to introduce the following definitions.

Definition 8.6. Say a function f is *C-analytic in D* if it is analytic in the connected open set D and continuous in some larger open set D' containing $\mathrm{cl}(D)$.

Definition 8.7. For a connected open set D, define

$$\partial_\infty D := \begin{cases} \partial\left(\mathrm{cl}(D)\right) & \text{if } D \text{ is bounded,} \\ \{\infty\} \cup \partial\left(\mathrm{cl}(D)\right) & \text{if } D \text{ is unbounded.} \end{cases}$$

An important point to note is that $\partial_\infty D$ does not equal ∂D if D is unbounded.

Lemma 8.8. *Let f be C-analytic in D. Suppose there exists a real constant K satisfying the property that $\limsup_{z\to a}|f(z)| \le M$ for all $a \in \partial_\infty D$. Then $|f| \le M$ throughout D.*

Proof. Let $D_\delta = \{z \in D : |f(z)| > M + \delta\}$ for any $\delta > 0$. We need to show that $D_\delta = \emptyset$. Suppose for contradiction that this is not the case. As $|f|$ is continuous, this implies D_δ is open in D. Furthermore, the points p on $\partial\left(\mathrm{cl}(D_\delta)\right)$ must have $|f(p)| = M + \delta$, so in fact $\mathrm{cl}(D_\delta) \subset D$ and is bounded, implying $\mathrm{cl}(D_\delta)$ is compact by the Heine-Borel theorem. Clearly f is analytic on D_δ as well. Thus $|f(z)| \le M + \delta$ on D_δ by the previous corollary, contradiction the definition of D_δ. □

Theorem 8.9 (Phragmén-Lindelöf)**.** *Let D be a connected and simply-connected open set with $\partial_\infty D$ equal to the disjoint union of exactly two connected components A and B (where $\{\infty\}$ is considered a connected component). Suppose there are C-analytic functions f and φ on D satisfying the following conditions:*

- *φ never vanishes and satisfies $|\varphi| \le K$ for some positive real constant K,*
- *there is a real constant M such that for every $a \in A$ and $b \in B$ and $N \in \mathbb{R}_{>0}$,*

$$\limsup_{z\to a}|f(z)| \le M \qquad \text{and} \qquad \limsup_{z\to b}|f(z)||\varphi(z)|^N \le M.$$

Then $|f| \le M$ throughout D.

Proof. As D is connected and simply-connected, corollary 7.8 tells us that we can define a single-valued branch of $\log\varphi$ on D. Thus we can define $g(z) = \exp(N\log\varphi(z))$ for any chosen $\eta > 0$, which is analytic on D and satisfies $|g| = |\varphi|^N \le K^N$. Define the analytic function $F : D \longrightarrow \mathbb{C}$ by

$$F(z) := f(z)g(z)K^{-N}.$$

Note that $|F| \le |f|$ by definition of g, and that $|F| = |f||g|^N K^{-N} = |f||\varphi|^N K^{-N}$. By the assumptions on f, we see that $|F| \le \max\{M, K^{-N}M\}$ using the previous lemma. This gives

$$|f| \le |\varphi|^{-N}K^N \max\{M, K^{-N}M\} = |\varphi|^{-N}\max\{K^N M, M\},$$

so, as $|\varphi| \ne 0$, we get the desired result after letting $N \to 0^+$. □

The above version of the Phragmén-Lindelöf theorem is found in chapter VI, theorem 4.1 of [3], and is the most general one out of all the sources cited in the bibliography. It gives rise to the following corollary, which subsumes results in other complex analysis books. In particular, the case $k = 1$ for part (b) of this next corollary is usually called the Phragmén-Lindelöf theorem, instead of the general one above.

Corollary 8.10. *Suppose f is C-analytic in D, and let M be a real constant.*

(a) Suppose $D = \{z \in \mathbb{C} : |\arg(z)| < \pi/(2k)\}$ for $k \ge 1/2$. Also suppose $|f(z)| \le M$ for every z lying on the rays $\partial = \{z \in \mathbb{C} : |\arg(z)| = \pi/(2k)\}$, and suppose that there exists a real constant A and $l \in (0, k)$ satisfying $|f(z)| \le A\exp(|z|^l)$ for $z \in D$ with $|z|$ sufficiently large. Then $|f| \le M$ throughout D.

(b) Suppose $D = \{z \in \mathbb{C} : |\arg(z)| < \pi/(2k)\}$ for $k \ge 1/2$. Also suppose $|f(z)| \le M$ for every z lying on the rays $\partial = \{z \in \mathbb{C} : |\arg(z)| = \pi/(2k)\}$, and suppose that for every $\epsilon > 0$ there exists a real constant A_ϵ satisfying $|f(z)| \le A_\epsilon \exp(\epsilon|z|^k)$ for $z \in D$ with $|z|$ sufficiently large. Then $|f| \le M$ throughout D.

(c) Suppose $D = \{z \in \mathbb{C} : -\alpha/2 < \arg z < \alpha/2\}$ for some fixed $0 < \alpha \le 2\pi$. Suppose $|f| \le M$ on ∂D, and suppose that for every $\epsilon > 0$ there exists a real constant A_ϵ satisfying $|f(z)| \le A_\epsilon \exp(\epsilon|z|^{\pi/\alpha})$ for $z \in D$ with $|z|$ sufficiently large. Then $|f| \le M$ throughout D.

Proof. (a) Choose $m \in (l, k)$, and let $\varphi(z) = \exp(-z^m)$. We are done if we can verify the assumptions for the Phragmén-Lindelöf theorem for this choice of φ. Clearly $\partial_\infty D$ is the disjoint union of two connected components, with one of them the point at infinity. Also, φ never vanishes. If we write $z = |z|e^{i\theta}$ using Euler's formula, we see that $|\varphi(z)| = \exp(-|z|^m \cos m\theta) \le 1$, since on D we have

$$0 < \rho := \cos\frac{m\pi}{2k} < \cos m\theta \le 1.$$

Next, continuity of f certainly tells us that $\limsup_{z\to a}|f(z)| \le M$ for all $a \in \partial$, and

$$\begin{aligned}|f(z)||\varphi(z)|^N &\le A\exp(|z|^l - |z|^m N\cos m\theta)\\ &\le A\exp|z|^m(|z|^{l-m} - N\cos m\theta)\\ &\le A\exp|z|^m(|z|^{l-m} - N\rho).\end{aligned}$$

As $l - m < 0$ and N, ρ are constants, we see at once that $\limsup_{z\to\infty}|f(z)||\varphi(z)|^N = 0 \le M$. This verifies all the assumptions in the Phragmén-Lindelöf theorem.

(b) Define the analytic function $F : D \longrightarrow \mathbb{C}$ by $F(z) = f(z)\exp(-\delta z^k)$, where $\delta > 0$ is arbitrary and fixed. Choosing $\epsilon \in (0, \delta)$, we see that

$$|F(x)| \le A_\epsilon \exp\left((\epsilon - \delta)x^k\right)$$

for any positive real x. As $\epsilon - \delta < 0$, by letting $x \to \infty$ we see that $|F(x)| \to 0$. Hence the positive real number $M_1 := \sup\{|F(x)| : x \in \mathbb{R}_{>0}\}$ exists and is finite. Let $M_2 := \max\{M, M_1\}$, and define

$$D_+ := \left\{z \in D : 0 < \arg z < \frac{\pi}{2k}\right\} \qquad \text{and} \qquad D_+ := \left\{z \in D : -\frac{\pi}{2k} < \arg z < 0\right\}.$$

Letting $\partial_+ := \partial(\mathrm{cl}(D_+))$ and $\partial_- := \partial(\mathrm{cl}(D_-))$, it is clear that $\limsup_{z\to b}|f(z)| \le M_2$ for any $b \in \partial_+ \cup \partial_-$. By doing an angle rotation of ∂_+ and ∂_- to the region $\{z \in \mathbb{C} : |\arg(z)| < \pi/(4k)\}$, part (a) then implies that $|F| \le M_2$ throughout D.

We now show that $M_2 = M$. If not, then necessarily $M_2 = M_1 > M$, so $|F|$ assumes a maximum value on $\mathbb{R}_{>0}$, implying F is constant by the maximum modulus principle (theorem 8.1). This implies $f(z) = c\exp(\delta z^k)$ for some constant c, contradicting the boundedness of f along ∂. Now, the equality $M_2 = M$ implies $|f(z)| \le M|\exp(\delta z^k)|$. As M is independent of δ, we are done by letting $\delta \to 0$.

(c) After normalization it suffices to assume $M = 1$. Consider the function g defined on the right half plane $H_r = \{z \in \mathbb{C} : \Re(z) > 0\}$ by $g(z) := f(z^{\alpha/\pi})$. Then $|g(z)| \le 1$ on the imaginary axis, and $|g(z)| = |f(z^{\alpha/\pi})| \le A_\epsilon \exp\left(\epsilon(|z|^{\pi/\alpha})^{\alpha/\pi}\right) = A_\epsilon e^{\epsilon|z|}$, so we are done by applying part (b) in case $k = 1$. □

There are many more Phragmén-Lindelöf-like theorems that can be shown using theorem 8.9. For example, one more will be given in the next section as a lemma in our discussion on convexity. To end this section, we demonstrate how to get some Liouville-like results using the above corollary.

Corollary 8.11. *Let f be a non-constant entire function.*

(a) There exists a curve along which f approaches infinity.

(b) Suppose that for each $\epsilon > 0$ there exists a real constant A_ϵ satisfying $|f(z)| \le A_\epsilon e^{\epsilon\sqrt{z}}$ throughout the complex plane. Then f is unbounded on every ray through the origin.

Proof. (a) Let us consider $T_1 := \{z \in \mathbb{C} : |f(z)| > 1\}$. From here we consider any connected component S_1 of T_1, which is open by continuity of f. The boundary $\partial(\mathrm{cl}(S))$ of $\mathrm{cl}(S)$ is contained in the boundary of $\mathrm{cl}(T)$. Hence $|f(b)| \ge 1$ for all $\partial(\mathrm{cl}(S))$ by continuity of f, and in fact $|f(b) = 1$ for it not to be in the interior. Therefore f is unbounded on S_1, else $f(\mathrm{cl}(S_1))$ is bounded by the Heine-Borel theorem, so part (a) of corollary 8.5 will tell us $|f| \le 1$ on S_1, a contradiction.

From here, iteratively choose $T_{k+1} = \{z \in S_k : |f(z)| > k+1\}$ and S_{k+1} a connected component of T_{k+1}. Notice T_{k+1} is nonempty by unboundedness of f along S_k the properties of S_1 still holds for S_{k+1}. Hence we have a decreasing sequence $S_1 \supset S_2 \supset \cdots$, and we are done by constructing a curve along this decreasing sequence of open sets.

(b) Suppose f were bounded on a ray R. Then part (c) of the previous corollary, with $\alpha = 2\pi$ and f appropriately normalized, tells us that f is also bounded on $\mathbb{C}\setminus R$. Liouville's theorem (corollary 4.8) would then tells us that f is constant, contradicting our assumption. □

9. The Picard theorems

We'll just state the theorems here and not bother with the proofs. See [3] for a reference.

Theorem 9.1 (Little Picard)**.** *If f is an entire function that omits more than one value on the complex plane, then f is a constant.*

Theorem 9.2 (Great Picard)**.** *Suppose f is an analytic function with an essential singularity at z_0. Let N be any neighborhood of z_0 inside the domain of f. Then f assumes values at each complex number, with at most one exception, an infinite number of times in N.*

10. Conformal maps

Definition 10.1. An analytic function $f : D_1 \longrightarrow D_2$ is *k-to-1* if the equation $f(z) = d$ has k roots for every $d \in D_2$, counted up to multiplicity. If f is an injective analytic function, then it is called a *conformal mapping*. Two connected open sets D_1 and D_2 are *conformally equivalent* if there exists a bijective analytic function f between them.

The rest of this section studies why we define conformal mappings in this way, and more importantly prove the Riemann mapping theorem. Before that, let us say what conformality actually means.

Definition 10.2. A function f defined in a neighborhood of z_0 is *conformal* at z_0 if f preserves angles there. In more detail, if C_1 and C_2 are curves intersecting at z_0 and locally smooth at z_0, then the angle of the tangents at C_1 and C_2 at z_0 is equal to the that of $f \circ C_1$ and $f \circ C_2$ at $f(z_0)$.

The proposition below explains our definition of conformal mappings.

Proposition 10.3. *Suppose f is analytic at z_0. Let $k := \min\{m \in \mathbb{Z} : f^{(m)}(z_0) \neq 0\}$.*
(a) If $f'(z_0) \neq 0$, then f is conformal and locally injective at z_0.
(b) Suppose $f'(z_0) = 0$ and f is nonconstant. Then f magnifies angles at z_0 by a factor of k, and f is a k-to-1 mapping in some sufficiently small open set containing z_0.

Proof. (a) Let $z(t) = x(t) + iy(t)$ be a smooth curve at $z_0 = z(t_0)$. Then the angle of inclination to the positive real axis is given by $\arg z'(t_0)$. By the chain rule, we have

$$\begin{aligned}\arg(f \circ z)'(t_0) &= \arg f'(z(t_0)) \cdot z'(t_0) \\ &= \arg f'(z_0) + \arg z'(t_0),\end{aligned}$$

where the last equality follows by Euler's formula and the fact that $f'(z_0) \neq 0$. Thus the function f adds the angle of inclination of any curve through z_0 by the constant $\arg f'(z_0)$, implyting conformality at z_0.

Next we show f is locally injective at z_0. Since f is analytic and f' is nonconstant as $f'(z_0) \neq 0$, this implies $g(z) := f(z) - f(z_0)$ is analytic and nonconstant at z_0 as well. Hence the zeros of g are isolated by the identity principle (corollary 5.4). Furthermore, by a Taylor series expansion of f at z_0, we see that z_0 is a zero of order one for g. Hence f is locally injective by part (b) of corollary 7.9.

(b) In this case, the Taylor series of f in a small disk about z_0 tells us that $f(z) = (z - z_0)^k g(z)$, where g is analytic and $g(z_0) \neq 0$. By corollary 7.8, the function $\sqrt[k]{g}$ is defined and analytic in this small disk. Hence, letting $h(z) := (z - z_0)\sqrt[k]{g(z)}$, it is clear that $f(z) = z^k \circ h(z)$. Now, $h'(z_0) \neq 0$, wo h is conformal and locally injective at z_0. Thus, as $h(z_0) = 0$, it remains to see that the analytic function $z \mapsto z^k$ magnifies angles at 0 by a factor of k, and is a k-to-1 mapping. But these two claims are all clear, the former using Euler's formula, and the latter using the fundamental theorem of algebra (corollary 4.9). ☐

After explaining the definition of a conformal mapping, the main goal of this section is to understand the Riemann mapping theorem. It will be very related to the important special mappings discussed in the next section.

Theorem 10.4 (Riemann mapping theorem)**.** *Let G be a connected and simply-connected open set that is not the whole complex plane. Then G is conformally equivalent to the open disk $\mathbb{D} := \operatorname{int}(D^2)$. Furthermore, for any $g \in G$ there exists a bijective analytic function $\alpha : G \longrightarrow \mathbb{D}$ such that $\alpha(g) = 0$ and $\alpha'(g) \in \mathbb{R}_{>0}$.*

Before proving this, let us look at one theoretical example and an application. We start by doing an example that is reminiscent of a corollary of the Phragmén-Lindelöf theorem (corollary 8.10).

Example 10.5. Suppose f is injective and analytic on the right half plane $H_r = \{z \in \mathbb{C} : \Re(z) > 0\}$, such that $f(H_r) \subset H_r$ and f fixes a point p. Then $|f'(p)| \leq 1$. Indeed, the Riemann mapping theorem gives us the existence of a bijective analytic function $\alpha : H_r \longrightarrow \mathbb{D}$ such that $\alpha(p) = 0$ and $\alpha'(p) \in \mathbb{R}_{>0}$, so we can consider the analytic function $\alpha \circ f \circ \alpha^{-1} : \mathbb{D} \longrightarrow \mathbb{D}$ that fixed 0. This function satisfies the conditions of Schwarz's lemma (corollary 8.4), so we see that

$$\begin{aligned} 1 &\geq \left|(\alpha \circ f \circ \alpha^{-1})'(0)\right| \\ &= \left|\alpha'\left((f \circ \alpha^{-1})(0)\right) \cdot f'\left(\alpha^{-1}(0)\right) \cdot (\alpha^{-1})'(0)\right| \\ &= \alpha'\left(f(p)\right) \cdot |f'(p)| \cdot (\alpha^{-1})'(0) \\ &= \alpha'\left(p\right) \cdot (\alpha^{-1})'(0) \cdot |f'(p)| \\ &= |f'(p)|\,, \end{aligned}$$

where the second last equality is by assumption that f fixes p, and the last equality is by corollary 7.10. Notice that we can actually replace H_r by any connected and simply connected open set in this example.

The corollary below is another criterion for us to analytically continue functions at removable singularities.

Corollary 10.6. *Let D be a connected open set and let $z_0 \in D$. Suppose that there is an analytic function $f : D \setminus \{z_0\} \longrightarrow \mathbb{C}$ such that its image set S is bounded. Then f has a removable singularity at z_0. Furthermore, if f is injective, then $f(z_0) \in \partial S$.*

Proof. Choose an open disk D' centered at z_0 and contained in D. The Riemann mapping theorem gives us a bijective analytic function $\alpha : D' \longrightarrow \mathbb{D}$ such that $\alpha(z_0) = 0$, so we can consider the map $g := \alpha \circ \alpha^{-1} \circ f$ from D' to $\mathbb{C}$. This map agrees with f at $D' \setminus \{z_0\}$, so we define the analytic continuation by setting

$$f(z) = \begin{cases} g(z) & \text{if } z \in D', \\ f(z) & \text{if } z \in D \setminus D'. \end{cases}$$

Note that $\operatorname{im} f$ is bounded, so by continuity $f(z_0)$ is defined. As before we write $f(z_0)$ to be the value of z_0 at this analytic continuation. If f is injective, then $f(z_0) \notin S$, but continuity forces us to have $f(z_0) \in \partial S$. ☐

As a quick application of the above corollary, notice that there is no bijective analytic function f from a punctured disk D_{punc} to an annulus A. Else this corollary gives us a homeomorphism from a simply connected space to a space with nontrivial fundamental group, contradicting the fact that homotopy equivalent spaces have isomorphic fundamental groups.

11. Interpreting Cauchy's theorems topologically

To consider the general form of Cauchy's theorem, we need hint on what singular homology is.

Definition 11.1. A *cycle* is a formal sum $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$, where each a_i are positive integers, and the γ_i are pairwise different closed curves in an open set D. The word different in the previous definition is in the sense that $\gamma_i \neq -\gamma_j$ for all i, j, and that $\int_{\gamma_i} f \neq \int_{\gamma_j} f$ for at least one function f. The integral of such a cycle is defined by

$$\int_\gamma f := \sum_{i=1}^n a_i \int_{\gamma_i} f.$$

Observe that any cycle γ in a connected and simply connected open set D satisfies $\eta(\gamma, z_0) = 0$ for all z_0 not in D. The general form of Cauchy's theorem describes cycles of this form. Recall that a cycle γ in an open set D is *homologous to zero* with respect to D if $\eta(\gamma, z_0) = 0$ for all z_0 in the complement of D (Definition 7.3).

Theorem 11.2 (Generalized Cauchy)**.** *If f is analytic in an open set D, then $\int_\gamma f = 0$ for every cycle γ which is homologous to zero in D.*

Let us note how this immediately generalized Cauchy's integral formula

Theorem 11.3 (Generalized Cauchy's integral formula)**.** *If f is analytic in an open set D, then*

$$f(z_0) = \frac{1}{2\pi i \eta(\gamma, z_0)} \int_\gamma \frac{f(z)}{z - z_0} dz$$

for every cycle γ which is homologous to zero in D.

Proof. The proof proceeds as before, but with some minor changes. Define, as before,

$$F(z) := \frac{f(z) - f(z_0)}{z - z_0}.$$

This is analytic away from z_0, and by proposition 5.1 we can analytically extend F to z_0 in a unique way. Hence F is defined on D, and the generalized Cauchy's theorem on closed curves implies $\int_\gamma F = 0$. This gives the formula after rearranging. □

Example 11.4. Let D be a simply-connected region. If f is analytic in D, then it is clear that $\int_\gamma f = 0$ for all cycles γ in D by the generalized Cauchy's theorem.

Example 11.5 (Independence of Homotopies)**.** The generalized Cauchy's theorem tells us that line integrals on analytic functions are independent of homotopies in the domain of definition, even for multiply-connected spaces (i.e. spaces that are not simply connected). For example, suppose f is analytic on an annulus inner and outer circles of radii r and R respectively. Let γ_1 and $-\gamma_2$ be two circles in the annulus, where the minus sign indicates the opposite orientation. Then the generalized Cauchy's theorem tells us that $\int_{\gamma_1 - \gamma_2} f = 0$ (implicitly using proposition 4.3 here), so in fact $\int_{\gamma_1} f = \int_{\gamma_2} f$.

We now prove the generalized Cauchy's theorem, following section 4.5 of [1] closely.

Proof of theorem 11.2. It suffices to assume that the open set D is bounded, since if it is unbounded then we can simply consider a bounded subspace containing γ. Let us cover the plane by a net of squares with sides of length δ, and let D_δ be the union of the squares in the interior $\mathrm{int}(D)$ of D. We choose $\delta > 0$ sufficiently small such that there is at least one square that is $\mathrm{int}(D)$, and that γ is in the interior of D_δ. Let S be one of the squares that makes up D_δ. Then Cauchy's integral formula (theorem 4.4) and Cauchy's theorem on rectangles (lemma 3.9) tells us that

$$\frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{if } z \in \mathrm{int}(D), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by adding up along all the squares making up D_δ, we see that

$$f(z) = \frac{1}{2\pi i} \int_{B_\delta} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where B_δ is the boundary of D_δ. Hence

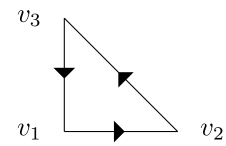
$$\begin{aligned} \int_\gamma f &= \int_\gamma \left(\frac{1}{2\pi i} \int_{B_\delta} \frac{f(\zeta)}{\zeta - z} d\zeta \right) dz \\ &= \int_{B_\delta} \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} dz \right) f(\zeta) d\zeta, \end{aligned}$$

where the order of integration can be reversed as the integrand is continuous in both integration variables. Hence inside integral in the final expression above equals $-\eta(\gamma, \zeta)$. It remains to show this equals zero. Indeed, as γ is contained in $\mathrm{int}(D)$ and $\zeta \in B_\delta$, there exists a path starting at ζ and ending at a point in the complement of D such that the path does not intersect γ. Hence we are done by applying part (b) of proposition 4.3 to each closed curve of γ. □

Let us end this section by remarking that definition 7.3 comes from the idea of singular homology. Recall that the group $C_n(D)$ of singular n-chains is defined to be the free abelian group on continuous chain maps $\sigma : \Delta^n \longrightarrow D$, where $\Delta^n := [v_1, \ldots, v_n]$ is the standard n-simplex. The differential from $C_2(D)$ to $C_1(D)$ is defined to be

$$\partial\sigma = \sigma|_{[v_2,v_3]} - \sigma|_{[v_1,v_3]} + \sigma|_{[v_1,v_2]},$$

which induces an orientation on the boundary of Δ^2 to make $\partial\sigma$ an oriented closed curve, as drawn below.



Continuing the idea of proposition 4.3, we observe the following proposition, which will not be needed for the rest of the notes. Our proof elaborates on the proof of proposition 1.9.13 in [2], and is fairly similar to the proof of the generalized Cauchy's theorem.

Proposition 11.6. *Let* $K := \operatorname{im}\left(C_2(D) \xrightarrow{\partial} C_1(D)\right)$*. Then a chain* γ *is homologous to zero if and only if it is in* K*.*

Proof. ($\Leftarrow$) Clearly, every element in K is homologous to zero as a continuous map is nullhomotopic if and only if its domain is contractible. In particular, the image of a singular 2-chain is contractible in D, so the bounded region it determines do not contain any element in the complement of D. Then use par (c) of proposition 4.3 to conclude.

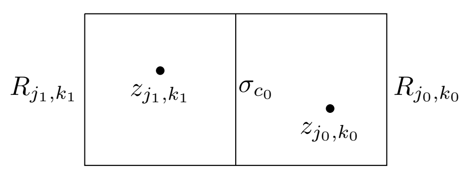
($\Rightarrow$) We want to show that every cycle γ that is homologous to zero is in K. By part (d) of proposition 4.3 and the fact that every closed curve can be locally approximated by a polygonal curve, we can assume every curve in γ can be thought of as an element $\sum_j c_j \nu_j$ in $C_1(D)$, where ν_j is either a horizontal or vertical segment. In particular, by letting $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be increasing sequences of real numbers, ν_j can be assumed to be of the form $[\alpha_j, \alpha_{j+1}] \times \{\beta_j\}$ or $\{\alpha_j\} \times [\beta_j, \beta_{j+1}]$ when viewed in the real plane.

We now consider the rectangles $R_{j,k} := [\alpha_j, \alpha_{j+1}] \times [\beta_k, \beta_{k+1}]$, and pick one $z_{j,k}$ in the interior of $R_{j,k}$. Consider the 1-cycle

$$\gamma_{approx} := \sum_{j,k} \eta(\gamma, z_{j,k}) \partial(R_{j,k}).$$

To see this is the boundary of the 2-chain $\sum_{j,k} \eta(\gamma, z_{j,k}) R_{j,k}$, we need to show that if $R_{j,k} \not\subset D$ then $\eta(\gamma, z_{j,k}) = 0$. But this is true by part (b) of proposition 4.3.

Finally, we need to show that $\gamma_{approx} = \gamma$, thus showing that γ is in K. Write $\gamma - \gamma_{approx} = \sum_j n_j \sigma_j$, where n_j are integers and σ_j are horizontal or vertical segments lying on the boundaries of $R_{j,k}$ with pairwise disjoint interiors. Inductively, we want to show that $\gamma_{c_0} := \gamma - \gamma_{approx} + n_{c_0} \partial(R_{j_0,k_0})$ is zero for some chosen index c_0 such that $\partial(R_{j_0,k_0})$ shares the edge σ_{c_0} with another neighboring rectangle R_{j_1,k_1} with z_{j_1,k_1} in its interior. The picture to have in mind is drawn below.



By definition of γ_{approx}, edges of γ that is not shared between more than two rectangles $R_{j,k}$ will be cancelled out by γ_{approx}, so each σ_j is shared between at least two of $R_{j,k}$. Notice that $\eta(\gamma - \gamma_{approx}, z_{j,k}) = 0$ as either $z_{j,k}$ were already in the unbounded regions determined by both γ and γ_{approx}, or $\gamma - \gamma_{approx}$ allowed this to happen. Also notice that $\eta(\partial(R_{j_0,k_0}), z_{j_0,k_0}) = 1$ trivially as $\partial(R_{j_0,k_0})$ is homotopic to a circle centered at z_{j_0,k_0}. Hence $\eta(\gamma_{c_0}, z_{j_0,k_0}) = 0$. Similarly, $\eta(\partial(R_{j_0,k_0}), z_{j_1,k_1}) = 0$ since it is outside of the region bounded by $\partial(R_{j_0,k_0})$, so $\eta(\gamma_{c_0}, z_{j_1,k_1}) = 0$. Furthermore, γ_{c_0} does not contain any multiple of the line segment σ_{c_0} since the term $n_{c_0}\sigma_{c_0}$ in $\gamma - \gamma_{approx}$ cancels the term $-n_{c_0}\sigma_{c_0}$ in $\partial(R_{j_0,k_0})$. Therefore there is a path from z_{j_1,k_1} to z_{j_0,k_0} that does not intersect γ, implying $n_{c_0} = 0$ by part (b) of proposition 4.3. We are done. $\square$

References

[1] Lars Ahlfors, *Complex Analysis (3rd Edition)*. McGraw-Hill Education, 1979.
[2] Carlos Berenstein and Roger Gay, *Complex Variables: An Introduction*. Graduate Text in Mathematics **125**, 1991.
[3] John Conway, *Functions of One Complex Variable*. Graduate Text in Mathematics **11**, 1978.
[4] Elias M. Stein and Rami Shakarchi , *Complex Analysis*. Princeton University Press, 2003.