$\mathbb{F}_q[\theta]$ is Level; ζ is You; Li is Win

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Abstract

This short expository note consists of two interconnected parts. The first part highlights some number-theoretic properties of the function field ring of integers $\mathbb{F}_q[\theta]$. The second part gives an overview of the Carlitz ζ-function, which is the function field analog of the Riemann ζ-function.

Everything here has analogs in the number field setting; see [3] for such a reference.

1 $\mathbb{F}_q[\theta]$ is Level

Let q be a power of a prime p. Then the ring $A = \mathbb{F}_q[\theta]$ is the polynomial ring in θ with coefficients lying in the finite field $\mathbb{F}_q$. This is the ring of integers for the function field $k = \mathbb{F}_q(\theta)$.

The purpose of the first section is to give quick proofs of the function field *abc*-Conjecture and Prime Number Theorem. Most of the exposition surrounding the *abc*-Conjecture is based on [1].

Theorem 1 (*abc*-Conjecture)**.** *Let $a, b, c \in A$ be coprime and satisfying the following conditions:*

- *$a + b = c$;*
- *if the factorization of c over $\overline{\mathbb{F}_q}$ has a factor of degree dividing p, then both a, b has no factors of degree dividing p over $\overline{\mathbb{F}_q}$;*
- *if the factorization of a or b over $\overline{\mathbb{F}_q}$ has a factor of degree dividing p, then c has no factors of degree dividing p over $\overline{\mathbb{F}_q}$.*

If abc has exactly k distinct zeros in $\overline{\mathbb{F}_q}$, then the degrees of each of the polynomials a, b, c cannot exceed $k-1$.

The *abc*-Conjecture is false is we remove the assumption of divisibility of factors; see Davenport's Theorem below for a counterexample when 3 divides q.

Proof of abc-Conjecture. Let $f = a/c$ and $g = b/c$. Then $f + g = 1$ and $f' = -g'$ by assumption. Now write

$$a = u\prod_{\alpha}(x-\alpha)^{r_\alpha}, \qquad b = v\prod_{\beta}(x-\beta)^{s_\beta}, \qquad c = w\prod_{\gamma}(x-\gamma)^{t_\gamma},$$

where $u, v, w \in \mathbb{F}_q^\times$, and α, β, γ are the roots of a, b, c. Because a, b, c are coprime, these roots are all distinct. Also, a computation reveals

$$\frac{f'}{f} = \sum_{\alpha}\frac{r_\alpha}{x-\alpha} - \sum_{\gamma}\frac{t_\gamma}{x-\gamma};$$
$$\frac{g'}{g} = \sum_{\beta}\frac{s_\beta}{x-\beta} - \sum_{\gamma}\frac{t_\gamma}{x-\gamma}.$$

By assumption, none of the two expressions above are zero. Hence, if we define

$$h = \prod_{\alpha}(x-\alpha)\prod_{\beta}(x-\beta)\prod_{\gamma}(x-\gamma),$$

then h is a degree k polynomial, and the functions

$$\phi = \frac{hf'}{f}, \qquad \psi = \frac{hg'}{g}$$

are polynomials of degrees at most $k-1$. Now, the relation $f' = -g'$ gives

$$b\psi = a\phi,$$

and we are done as a and b are coprime, and $c = a + b$. $\square$

Example 2 (Fermat's Last Theorem)**.** The *abc*-Conjecture gives a quick proof of the nonexistence of solutions to

$$f^n + g^n = h^n, \qquad f, g, h \in A,$$

for $n \geq 3$ and $p \nmid n$. If there were such a solution, then we can assume f, g, h are coprime. Hence, the *abc*-Conjecture implies

$$n \deg f \leq \deg f + \deg g + \deg h - 1,$$

and similarly by replacing the left hand side with $n \deg g$ and $n \deg h$. Hence, by summing the three inequalities,

$$n(\deg f + \deg g + \deg h) \leq \deg f + \deg g + \deg h - 1,$$

and this is only possible when $n \leq 3$. Note that the existence of solutions is trivial for $n = 1$, and for $n = 2$ such a solution is given by

$$f = x^2 - 1, \qquad g = 2x, \qquad h = x^2 + 1.$$

Note that this looks a lot like the rational Pythagorean triples. In fact, the argument over there applies here to show that all (primitive) Pythagorean triples in our case are of the form

$$f = c(u^2 - v^2), \qquad g = 2uv, \qquad h = c(u^2 + v^2),$$

where $c \in \mathbb{F}_q^\times$ and $u, v \in A$ are relatively prime.

Example 3 (Davenport's Theorem)**.** Davenport's Theorem is the following statement: If $3 \nmid q$, and f and g are coprime nonconstant polynomials, then

$$\deg(f^3 - g^2) \geq \frac{\deg f}{2} + 1.$$

Note that this bound cannot be improved since equality is attained with $f = x^2 + 2$ and $g = x^3 + 3x$, giving $f^3 - g^2 = 3x^2 + 8$. (This example also tells us the above inequality is false if 3 divides q.)

In order to prove Davenport's Theorem, first note that the inequality is trivial if $\deg f^3 \neq \deg g^2$, so we can assume $\deg f^3 = \deg g^2$. Let $\deg f = 2m$ and $\deg g = 3m$. Now

$$(f^3 - g^2) + g^2 = f^3,$$

and the terms are pairwise coprime by assumption. Also, the zeros of $(f^3 - g^2)g^2 f^3$ cannot exceed the sum $\deg(f^3 - g^2) + \deg g + \deg f$. Thus, by the *abc*-Conjecture,

$$\deg g^2 \leq \deg(f^3 - g^2) + \deg g + \deg f - 1,$$

implying

$$\deg(f^3 - g^2) \geq m + 1 = \frac{\deg f}{2} + 1,$$

as desired.

There are other applications of the *abc*-Conjecture (e.g. Catalan's Equation), and computations of some solutions to Pell's Equations, in [1]. One can also find more information about Pell's Equation for polynomials in the literature (we can solve this just like in the classical case).

We next prove the Prime Number Theorem. This is so named as it looks like the classical version:

$$\frac{q^n}{n} = \frac{x}{\log_q x}, \qquad \text{where } x = q^n.$$

Theorem 4 (Prime Number Theorem)**.** *Let N_n be the number of monic irreducible polynomials of degree n. Then*

$$N_n \sim \frac{q^n}{n}.$$

Proof. Let $F = \mathbb{F}_{q^n}$. Then every nonzero element in F is the root of a monic irreducible polynomial of degree dividing n, and so

$$q^n = |F| = \sum_{d|n} dN_d.$$

Now, applying Möbius inversion to the functions $A(d) = q^d$ and $B(d) = dN_d$, one gets

$$nN_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d,$$

where μ is the Möbius function. We are now done by a trivial bound. $\square$

Note that, by studying the proofs, the *abc*-Conjecture actually holds for all polynomial rings over a field, but not the Prime Number Theorem.

2 ζ is You; Li is Win

The second section will focus on the big picture, and is intentionally vague to avoid details. See [2] and the references there for some details (many more references are needed for full details).

As hinted in the title of this section, we can define a function field version of the Riemann ζ-value, called the *Carlitz ζ-value*, by

$$\zeta(n) = \sum_{a \in A_+} \frac{1}{a^n}, \qquad n \in \mathbb{Z}_{\geq 1}.$$

Here A_+ is the set of monic polynomials in $\mathbb{F}_q[\theta]$, and $\zeta(n)$ is an element in $k_\infty = \mathbb{F}_q((\frac{1}{\theta}))$, the completion of k at the infinite place of A. This ζ-value enjoys a number of good properties; for example, it is convergent on the stated interval (using the absolute value $|\theta| = q$ on k_∞), and there is still an Euler factor expansion

$$\zeta(n) = \prod_{p \text{ monic irreducible}} \frac{1}{1 - p^{-n}}.$$

However, we still do not know if we can analytically continue this, nor do we know of a functional equation! This is in contrast to the case of the Riemann ζ-function.

Remark. Of course, we can generalize the above to multizeta values, but we will not do that in this talk for simplicity. One way to guess for the definition of the multizeta values is to compute $\zeta(n)\zeta(m)$ and derive a shuffle relation via the inclusion-exclusion principle.

Note that the definition of the Carlitz ζ-value is defined over the sum of all monic polynomials. Why do we not sum over all polynomials? A reason is to preserve the analogy with the Riemann ζ-value. Recall in this classical case we took the sum over all positive integers. This is half of the nonzero elements in the ring of integers $\mathbb{Z}$, and an interpretation is that we took a representative among the nonzero integers with relation defined by $\mathbb{Z}^\times = \{\pm 1\}$. Similarly, in the function field case, we pick a representative among the nonzero polynomials with relation defined by $\mathbb{F}_q[\theta]^\times = \mathbb{F}_q^\times$, giving rise to our definition of $\zeta(n)$.

Here is another simpler reason for this definition: since

$$\sum_{c \in \mathbb{F}_q^\times} c^k = \begin{cases} 0 & \text{if } q-1 \nmid k \\ -1 & \text{if } q-1 | k \end{cases}$$

the sum $\sum_{a \in A} a^{-n}$ will be a terrible definition for $\zeta(n)$!

Because of the sum above, we define a positive integer n to be "even" if it divides $q-1$, and "odd" otherwise. Then Carlitz did the following computation.

Proposition 5. *For any "even" positive integer s,*

$$\zeta(s) = \frac{B_s}{\Gamma_{s+1}} \tilde{\pi}^s,$$

where B_s and Γ_{s+1} are the function field Bernoulli and Gamma numbers, and $\tilde{\pi}$ is the Carlitz period. $\square$

The number $\tilde{\pi}$ is the function field analog of π, and is an element in the algebraic closure of k_∞. The definition comes from considering the kernel of the exponential map of the Carlitz module (which is a Drinfeld module). Similarly, B_s and Γ_{s+1} are defined by a power series expansion using this exponential map.

Just as in the number field case, the values $\zeta(n)$ for n "odd" seems hard to compute. If we play around the values $\zeta(n)$, it is easy to get

$$\zeta(pn) = \zeta(n)^p,$$

but it seems like we can't write more down. We now have a basic question on ζ-values: can we understand all the algebraic relations between the values $\zeta(n)$? This question is notoriously difficult in the number field case (for example, we don't even know if $\zeta_R(3)$ is transcendental), but we have a complete answer for this in the function field case!

Theorem 6. *Each of the values $\zeta(1), \zeta(2), \ldots$ is transcendental over k. Furthermore, the set*

$$\{\tilde{\pi}, \zeta(n) : q-1 \nmid n \text{ and } p \nmid n\}$$

is algebraically independent over $\overline{k}$, and all algebraic relations between the values $\zeta(1), \zeta(2), \ldots$ are generated by the relations

$$\left\{\zeta(s) = \frac{B_s}{\Gamma_{s+1}} \tilde{\pi}^s : q-1|s\right\} \cup \{\zeta(pl) = \zeta(l)^p : l \in \mathbb{Z}_{\geq 1}\}.$$ $\square$

The proof of this uses the period interpretation of the Carlitz ζ-value. This is the correct way to think about this, and there are two reasons: correct analogy with the number field case, and allows generalization to multizeta values (for example, to be able to use in my thesis work). Without going into any details, here are the steps to proving the Theorem.

(1) Construct a function field analog of the polylogarithm, and write $\zeta(n)$ as a k-linear combination of these polylogarithms.
(2) Realize the polylogarithms and ζ-values as periods of uniformizable t-motives (those arising from successive extensions of tensor products of the Carlitz module).
(3) Study the motivic Galois group of these t-motives (which exists by Tannakian formalism) using difference equations, and relate this group to algebraic independence of polylogarithms.
(4) Show how this implies algebraic independence of the ζ-values.
(5) Using known algebraic relations, show how all these steps imply the Theorem above.

Remark. For multizeta values, the expectation is that a similar theorem should hold for all $\zeta(s_1, \ldots, s_r)$ with each s_i not divisible by $q-1$ and p. However, although we can still prove transcendence, we have only been able to prove algebraic independence for arbitrarily large families of them. This is due to the difficulty of the last two steps above, and the existence of many non-trivial relations between multizeta values.

Final Thoughts

The slogan is that many familiar notions in analysis and geometry usually carry over to the function field setting, though it seems to be more algebraic and combinatorial in nature. It is also usually easier to work with function fields than number fields. Here is a list of things hinted throughout the talk.

- Analysis: nonarchimedean analysis (e.g. rigid analytic functions, Fourier series, Hadamard products)
- Geometry: Drinfeld modules, t-motives, comparison theorems, rigid analytic spaces
- Special Functions: logarithms, multizeta values, modular forms

References

[1] Edward Barbeau, *Polynomial Excursions*. AARMS Lecture Notes, 2009.

[2] Chieh-Yu Chang and Jing Yu, *Determination of algebraic relations among special zeta values in positive characteristic*. Adv. Math. **216** (2007), no. 1, 321–345.

[3] Michel Waldschmidt, *Lectures on Multiple Zeta Values*. Special year in number theory at IMSc, 2011.