Equally-distributed-equivalent income from a number-theoretic viewpoint

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Abstract

The Atkinson index is a measure of income inequality, first defined in 1970 by invoking the idea of Equally Distributed Equivalent (EDE), i.e. the level of income that, if equally distributed in a hypothetical scenario, would give the same level of welfare as the current income distribution. Using functional analytic techniques with an L-function viewpoint, we reinterpret the Atkinson index as a harmonic-like weighted sum. This allows us to derive a duality principle on inequality that, among other things, implies minimizing poverty is equivalent to maximizing wealth at inequality aversion parameter $\epsilon = 2$. Furthermore, our reinterpretation allows us to use the Riesz-Fréchet Representation Theorem to broaden on the Pigou-Dalton Principle for EDE. We also explain an application to estimate the optimality of resource allocation towards achieving maximal welfare under equity considerations.

JEL classification: C65; D63; I14

1 Introduction

Many measures exist to estimate population inequality in both economics and health [3], with the Gini index being the most famous. However, a good measure for inequality analyses should satisfy three important properties [7]: subgroup decomposability, where total inequality is divided into its constituent components; the Pigou-Dalton Principle, where a transfer of a desirable variable (e.g. wealth) from the rich to the poor results in less inequality as long as it does not bring the rich to a worse situation than the poor; and avoids value judgement, for instance by including an explicit parameter that changes the weights placed on various percentiles of the distribution (to allow sensitivity analysis with respect to this parameter for policy decisions). Using this definition of goodness, the Gini index does not qualify as a good measure for inequality analyses as it does not avoid value judgement. Rather, a detailed analysis of inequality measures (by the same paper [7]) concluded that the Atkinson index [1] stands out as the best inequality measure in health. Additionally, a quantitative analysis of income inequality measures by Shorrocks imply the generalized entropy index [11, Equation 31] is the only one-parameter family that is subgroup decomposable under relatively weak restrictions on homogeneity, and the generalized entropy index can be viewed as a monotonic transformation of the Atkinson index. As such, we focus on the Atkinson index in this paper.

Atkinson defined his index in [1] by invoking the idea of Equally Distributed Equivalent (EDE), reflecting the willingness to trade off aggregate benefits for income to be more equally distributed. This index is defined on the real line as

$$EDE(\epsilon) := \begin{cases} \left(\sum_i H_i^{1-\epsilon}\mu_i\right)^{\frac{1}{1-\epsilon}} & \epsilon \neq 1 \\ \prod_i H_i^{\mu_i} & \epsilon = 1 \end{cases} \tag{1}$$

where H_i is the income level for subgroup i with each H_i distinct and nonzero, μ_i is a weight for subgroup i (with the sum of all μ_i equaling 1), and ϵ is the Atkinson inequality aversion parameter. The weights μ_i are chosen depending on the application. In Atkinson's original formulation, μ_i is simply the proportion of the total population at income level H_i, but μ_i can also be income-dependent, causality-dependent, and so on [10].

Our definition of $EDE(\epsilon)$ is the non-normalized form of the definition given by Atkinson. Also, instead of restricting ϵ to be strictly non-negative, we allow use of the entire real line as it gives rise to a duality principle (Duality Principle 5.1) with important applications in both economics and health.

The goal of this paper is to understand the Atkinson index as weighted sums of the income levels H_i. In order to agree with the original work of Atkinson, such a weighted sum has to give greater weight to lowest income at higher inequality aversion ϵ, implying that the weighted sum needs to be harmonic-like as it emphasizes sensitivity to low-income groups. Temporarily suppose $\epsilon > 2$ and abstractly consider the sum

$$AH(\epsilon) := \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

where $f_{i,\epsilon}$ are non-negative functions in ϵ to be determined. We will show the existence of unique $f_{i,\epsilon}$'s minimizing $AH(\epsilon)$ with respect to a technical constraint (Theorem 2.1). Furthermore, this minimum value coincides with the Atkinson index $EDE(\epsilon)$. For $\epsilon < 2$, we will show that $AH(\epsilon)$ can be analytically continued to the entire real line except for a removable singularity at $\epsilon = 1$. This implies the initial supposition of $\epsilon > 2$ can be lifted, and $AH(\epsilon)$ gives a way to decompose $EDE(\epsilon)$ into subgroup components for all values of ϵ.

The decomposition of the Atkinson index into $AH(\epsilon)$ will be proven in Section 2. Following this, we demonstrate how our techniques broaden on the Pigou-Dalton Principle for the Atkinson index, as well as giving greater subgroup-level insights in EDE weighting of subgroups (Section 3). We then illustrate a use of our decomposition in the context of resource allocation (Section 4), before ending with an important duality principle that unifies opposite metrics such as wealth and poverty (Section 5). Our duality principle is derived by showing that the harmonic-like sum $AH(\epsilon)$ is transformed into an arithmetic-like sum at $\epsilon \leq 0$, vastly generalizing a recent observation of Sterck [16] that minimizing overall poverty level is equivalent to maximizing EDE income at $\epsilon = 2$.

2 Functional analysis on Atkinson's index

The definition of Atkinson's index is a little opaque for subgroup-level analyses at first glance. For instance, it does not inform us how the H_i's are differentially weighted at certain Atkinson parameter ϵ. While Shorrocks performed a subgroup decomposition on the Atkinson index into an arithmetic-like sum to obtain subgroup-level information [11, 12], we decompose $EDE(\epsilon)$ in an alternative way as a harmonic-like sum

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

for a family of functions $f_{i,\epsilon}$ in ϵ and depending on H_i and μ_i. These functions $f_{i,\epsilon}$ are uniquely determined, and an expression can be derived by performing partial differentiation with respect to H_i. However, partial differentiation does not give much insight beyond the work done by Shorrocks in his analysis. In particular, Shorrocks mentioned that his analog of $f_{i,\epsilon}\mu_i$ cannot be reasonably considered to be weights, as their sum over the subgroups does not equal 1 in general.

It is worth mentioning that our proposed sum is harmonic-like while Shorrocks used an arithmetic-like sum in his analysis. We believe a harmonic-like sum is more appropriate as the Atkinson index is more sensitive to lower income subgroups as ϵ increases, which is reflected in an harmonic-like sum (but not an arithmetic-like sum). However, arithmetic-like sums do come in play in our analysis as they will be essential when considering negative metrics such as poverty level (Section 5).

We now list some basic properties of the Atkinson index $EDE(\epsilon)$ (Equation 1) that is almost immediate from definition. $EDE(\epsilon)$ gives increasingly greater weight to the subgroup with lowest income as ϵ increases, and is a monotonic decreasing function in ϵ. In particular, $EDE(\epsilon)$ plateaus to the minimum income level among the subgroups as ϵ increases, and plateaus to the maximum income level among the subgroups as ϵ decreases. Further, by definition of the utility function for Atkinson's index

$$U(x) = \frac{x^{1-\epsilon}}{1-\epsilon}$$

and its concavity, the Atkinson index satisfies the main properties of social welfare measures, such as income homogeneity, population homogeneity, and the Pigou-Dalton Principle [1, 11]. These three properties will also be elaborated on in the next Section.

The purpose of this section is to prove the following mathematical results. For convenience in the arguments, these results are formulated as the inverse of the Atkinson index as stated in Equation 1.

Theorem 2.1. *Let $\epsilon > 2$, let $p = \epsilon - 1$, and let q be the number such that $1/p + 1/q = 1$. Then there exist a unique collection of non-negative values $f_{i,\epsilon}$ satisfying*

$$\sum_i f_{i,\epsilon}^q \mu_i = 1,$$

and such that these $f_{i,\epsilon}$ maximizes the expression

$$F(f_{i,\epsilon}) := \sum_i \frac{f_{i,\epsilon}\mu_i}{H_i}.$$

Furthermore, this maximum value equals $EDE(\epsilon)^{-1}$.

Corollary 2.2. *The values $f_{i,\epsilon}$ in Theorem 2.1 can be explicitly determined:*

$$f_{i,\epsilon} := \left(\sum_j \left(\frac{H_j}{H_i} \right)^{1-\epsilon} \mu_j \right)^{-\left(1 + \frac{1}{1-\epsilon}\right)} = \left(\frac{EDE(\epsilon)}{H_i} \right)^{\epsilon - 2}.$$

This can be viewed as an analytic function at $\epsilon \neq 1$.

Theorem 2.3. *Let $\epsilon > 2$. Suppose $q > 1$ is a number satisfying the following two conditions.*

- *There exists unique values $f_i > 0$ such that*

$$\sum_i f_i^q \mu_i = 1$$

  and f_i maximizes the expression

$$F(a_i) := \sum_i \frac{1}{H_i} a_i \mu_i.$$

- *$F(f_i)$ equals $EDE(\epsilon)^{-1}$, where f_i is specified in the above condition.*

Then q is unique, and equals the value in Theorem 2.1.

Theorem 2.4 (Analytic Continuation)**.** *Let q be as defined in Theorem 2.1. Consider* *the real-valued function*

$$L(H_i, \epsilon) := \max_{\substack{\sum_i f_{i,\epsilon}^q \mu_i = 1 \\ f_i \geq 0}} \left\{ \sum_i \frac{f_{i,\epsilon}\mu_i}{H_i} \right\}$$

which is well-defined on $\epsilon > 2$ by Theorem 2.3. Then $L(H_i, \epsilon)$ can be analytically continued to the entire real line, except for a removable singularity at $\epsilon = 1$. Furthermore, $L(H_i, \epsilon)$ is a positive function with $L(H_i, \epsilon) = EDE(\epsilon)^{-1}$.

Theorem 2.5 (Functional Equation)**.** *The function $L(H_i, \epsilon)$ defined in Theorem 2.4 satisfies*

$$L(H_i, -\epsilon)^{-1} = L(H_i^{-1}, 2 + \epsilon).$$

In summary, the five mathematical statements above narrate the following story. Theorem 2.1 gives a mathematical justification that a harmonic-like decomposition for the Atkinson index at $\epsilon > 2$ is in fact robust. Corollary 2.2 explicitly defines the components $f_{i,\epsilon}$ of the harmonic-like decomposition. Theorem 2.3 tells us that the technical condition imposed on $f_{i,\epsilon}$ allows us to deform Atkinson weights in a well-defined manner as ϵ varies, and is the only sensible deformation that naturally extends Atkinson weights as ϵ varies. In particular, this Theorem shows our definition of EDE-factors in the next Section makes sense (Definition 3.1). Theorem 2.4 mathematically justifies our harmonic-like decomposition agrees with the Atkinson index at all ϵ (not just $\epsilon > 2$). Finally, Theorem 2.5 exhibits a dual pairing on the harmonic-like decomposition that results in a duality principle that will be explained in Section 5 (Duality Principle 5.1).

Proof of Theorem 2.1. Let X be a countable measure space with discrete probability measure μ, and let F be an injective real-valued positive Lebesgue-measurable function on X. Then the p-norm of F is simply

$$\|F\|_p = \left(\sum_i F_i^p \mu_i \right)^{\frac{1}{p}}.$$

In our case where $F_i = 1/H_i$,

$$\|F\|_p = \left(\sum_i H_i^{-p} \mu_i \right)^{\frac{1}{p}}.$$

By the Riesz-Fréchet Representation Theorem for L^p-spaces [15, Chapter 1], there exist non-negative values f_i such that

$$\|F\|_p = \max_{\substack{\sum_i f_i^q \mu_i \leq 1 \\ f_i \geq 0}} \left\{ \sum_i \frac{1}{H_i} f_i \mu_i \right\},$$

where q is the number such that $1/p + 1/q = 1$. If we can find values f_i for this equality to hold, then we are done as the left-hand side $\|F\|_p$ equals $EDE(\epsilon)^{-1}$ after recalling $p = \epsilon - 1$.

Following this discussion, we need to solve an optimization problem: Find numbers $f_i \geq 0$ that maximizes

$$\sum_i \frac{1}{H_i} f_i \mu_i \tag{2}$$

subject to the condition

$$\sum_i f_i^q \mu_i = l, \qquad 0 \leq l \leq 1. \tag{3}$$

Clearly, f_i cannot be simultaneously zero for all i. Using Lagrange Multipliers, there exist a constant λ_l depending on l such that

$$\frac{1}{H_i} \mu_i = \lambda_l q f_i^{q-1} \mu_i.$$

Hence

$$f_i = \left(\frac{1}{\lambda_l q H_i} \right)^{\frac{1}{q-1}}.$$

Substituting f_i to Equations 2 and 3 gives

$$\sum_i \frac{1}{H_i} f_i \mu_i = \lambda_l^{-\frac{1}{q-1}} \sum_i \left(\frac{1}{q H_i^q} \right)^{\frac{1}{q-1}} \mu_i$$

and

$$\lambda_l^{-\frac{1}{q-1}} = l^{\frac{1}{q-1}} \left(\sum_i \left(\frac{1}{q H_i} \right)^{\frac{q}{q-1}} \mu_i \right)^{-\frac{1}{q}}$$

Since q, H_i, μ_i are all known constants, to maximize Equation 2, we will need to maximize $\lambda_l^{-\frac{1}{q-1}}$, which requires maximizing $l^{\frac{1}{q-1}}$, and this last expression is an increasing function on l as $q-1>0$. Therefore, necessarily $l=1$ for our optimization problem.

In summary, for each f_i,

$$
\begin{aligned}
f_i &= \left(\frac{1}{\lambda_1 q H_i}\right)^{\frac{1}{q-1}} \\
&= \left(\frac{1}{qH_i}\right)^{\frac{1}{q-1}} \left(\sum_j \left(\frac{1}{qH_j}\right)^{\frac{q}{q-1}} \mu_j\right)^{-\frac{1}{q}} \\
&= \left(\sum_j \left(\frac{H_i}{H_j}\right)^{\frac{q}{q-1}} \mu_j\right)^{-\frac{1}{q}}. \qquad (4)
\end{aligned}
$$

These f_i's are exactly the $f_{i,\epsilon}$ we seek. □

Proof of Corollary 2.2. This is an algebraic manipulation of Equation 4 to rewrite it into two different ways. □

Proof of Theorem 2.3. By manipulating Equation 4,

$$
f_i = \left(\frac{EDE(k)}{H_i}\right)^{-\frac{1}{q-1}}, \qquad k = 2 + \frac{1}{q-1}.
$$

Therefore, we can view

$$
H(q) := \sum_i \frac{1}{H_i} f_i \mu_i
$$

as a function in q. To prove the Theorem, it suffices to show that this function is monotone decreasing in q. A computation tells us that the derivative with respect to q is

$$
\begin{aligned}
\frac{d}{dq} H(q) &= \sum_i \frac{1}{H_i} f_i' \mu_i \\
&= \frac{1}{(q-1)^2} \sum_i \frac{f_i \mu_i}{H_i} \ln\left(\frac{EDE(k)}{H_i}\right) + \frac{1}{(q-1)^3} \sum_i \frac{f_i \mu_i}{H_i} \frac{EDE'(k)}{EDE(k)} \\
&= -\frac{1}{q-1} \sum_i \frac{f_i \mu_i}{H_i} \ln(f_i) + \frac{1}{(q-1)^3} \sum_i \frac{f_i \mu_i}{H_i} \frac{EDE'(k)}{EDE(k)}. \qquad (5)
\end{aligned}
$$

Here, $EDE'(k)$ is the derivative of $EDE(k)$ with respect to q.

We now need to show that the first and second term of Equation 5 are both negative. The second term is negative as $EDE'(k)$ is the only part of the term that is negative:

$$EDE'(k) = EDE(k) \cdot \left(\frac{1}{(1-k)^2} \left(\sum_i H_i^{1-k} \mu_i \right) + \left(\sum_i H_i^{-k} \mu_i \right) \right) \cdot \frac{-1}{(q-1)^2}.$$

For the first term, note that

$$\sum_i \frac{f_i \mu_i}{H_i} \ln(f_i) = \ln \left(\prod_i f_i^{\frac{f_i \mu_i}{H_i}} \right)$$

so we are reduced to showing that the product inside the logarithm is at least 1. Using the weighted power mean inequality, more specifically the weighted GM-HM inequality [2, Chapter 3],

$$\prod_i f_i^{\frac{f_i \mu_i}{H_i}} \geq \left(\left(\frac{\sum_i \frac{f_i \mu_i}{H_i} f_i^{-1}}{\sum_i \frac{f_i \mu_i}{H_i}} \right)^{-1} \right)^{\sum_i \frac{f_i \mu_i}{H_i}}. \tag{6}$$

We now make the observation that

$$\sum_i \frac{f_i \mu_i}{H_i} f_i^{-1} = \sum_i \frac{1}{H_i} \cdot 1 \cdot \mu_i$$

so by the first condition in the statement of the Theorem,

$$\sum_i \frac{f_i \mu_i}{H_i} f_i^{-1} \leq \sum_i \frac{f_i \mu_i}{H_i}.$$

Hence the fraction in the right hand side of Equation 6 is at most 1, implying

$$\left(\left(\frac{\sum_i \frac{f_i \mu_i}{H_i} f_i^{-1}}{\sum_i \frac{f_i \mu_i}{H_i}} \right)^{-1} \right)^{\sum_i \frac{f_i \mu_i}{H_i}} \geq 1$$

and we are done. □

Proof of Theorem 2.4. Note that both $L(H_i, \epsilon)$ and $EDE(\epsilon)^{-1}$ are analytic functions defined on the interval $(-\infty, 1) \cup (1, \infty)$. As $L(H_i, \epsilon) = EDE(\epsilon)^{-1}$ on $(2, \infty)$, the Identity Theorem [6, Chapter 1] implies they must also be equal on $(1, \infty)$.

We now apply the Identity Theorem on $(-\infty, 1)$ by showing that $L(H_i, 1/n)$ equals

$EDE(1/n)^{-1}$ for all positive integers $n \geq 2$. By Corollary 2.2,

$$\begin{aligned} L\left(H_i, \frac{1}{n}\right) &= \sum_i \left(\sum_j \left(\frac{H_j}{H_i}\right)^{1-\frac{1}{n}} \mu_j \right)^{-1+\frac{1}{1-\frac{1}{n}}} \frac{\mu_i}{H_i} \\ &= \left(\sum_j H_j^{1-\frac{1}{n}} \mu_j \right)^{-\left(1+\frac{n}{n-1}\right)} \left(\sum_i H_i^{1-\frac{1}{n}} \mu_i \right) \\ &= EDE\left(\frac{1}{n}\right)^{-\left(1-\frac{1}{n}\right)\left(1+\frac{n}{n-1}\right)} EDE\left(\frac{1}{n}\right)^{1-\frac{1}{n}} \\ &= EDE\left(\frac{1}{n}\right)^{-1}, \end{aligned}$$

as desired. □

Proof of Theorem 2.5. If $\epsilon > 0$, a calculation reveals

$$\begin{aligned} L(H_i^{-1}, 2+\epsilon) &= \sum_j \sum_i H_j \left(\left(\frac{H_j}{H_i}\right)^{-(1+\epsilon)} \mu_i \right)^{-\left(1-\frac{1}{1+\epsilon}\right)} \mu_j \\ &= \left(\sum_j H_j^{1+\epsilon} \mu_j \right) \left(\sum_i H_i^{1+\epsilon} \mu_i \right)^{-\left(1-\frac{1}{1+\epsilon}\right)} \\ &= \left(\sum_i H_i^{1+\epsilon} \mu_i \right)^{\frac{1}{1+\epsilon}} \\ &= EDE(-\epsilon) \\ &= L(H, -\epsilon)^{-1} \end{aligned}$$

where the last equality is due to Theorem 2.1. Finally, the condition $\epsilon > 0$ can be dropped by the Identity Theorem. □

3 EDE-factors

Due to Section 2, the Atkinson index can be decomposed as a harmonic-like sum

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

where $f_{i,\epsilon}$ is defined as

$$f_{i,\epsilon} := \begin{cases} \left(\sum_j \left(\frac{H_j}{H_i} \right)^{1-\epsilon} \mu_j \right)^{-\left(1+\frac{1}{1-\epsilon}\right)} & \epsilon \neq 1 \\ \sum_j \ln \left(\frac{H_j}{H_i} \right) \mu_j & \epsilon = 1. \end{cases}$$

These $f_{i,\epsilon}$ satisfy a technical weighted normality condition:

$$\sum_i f_{i,\epsilon}^q \mu_i = 1, \qquad q = \frac{\epsilon - 1}{\epsilon - 2},$$

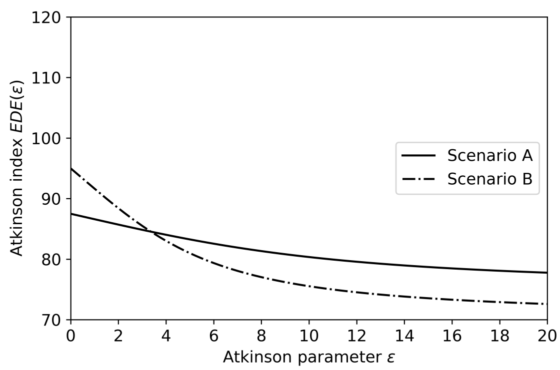
allowing us to extend Atkinson's weights μ_i through the expressions $f_{i,\epsilon}^q \mu_i$. In other words, an extension of Atkinson's weights is done via a q-analog of the expression $f_{i,\epsilon}\mu_i$. Such an extension requires agreement with Atkinson's weight without inequality aversion considerations, i.e. $w_i(0) = \mu_i$. As such, we need to shift ϵ by 1 for this equality to hold. Note that this is a choice of normalization to recover Atkinson's weigh at $\epsilon = 0$, and is not something intrintic to inequality aversion itself.

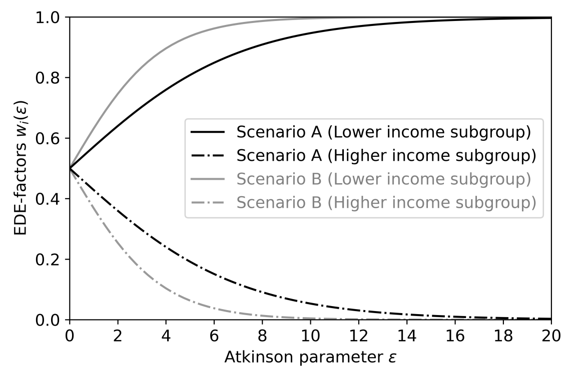
Definition 3.1. The *EDE-factor* for subgroup i can be explicitly defined in two ways:

$$w_i(\epsilon) := \left(\sum_j \left(\frac{H_j}{H_i} \right)^{-\epsilon} \mu_j \right)^{-1} \mu_i = \left(\frac{EDE(\epsilon + 1)}{H_i} \right)^{\epsilon} \mu_i.$$

Before discussing properties of EDE-factors, we give a simple example of EDE-factors on two hypothetical policy scenarios.

Example 3.2. Consider a population split into 2 subgroups with equal Atkinson weights $\mu_1 = \mu_2 = 0.5$ and income levels $H_1 = \$70,000$, $H_2 = \$100,000$. Also consider two Scenarios: A, where individuals in the lower income subgroup 1 are given additional $\$5,000$; and B, where individuals in the higher income subgroup 2 are given additional $\$20,000$. Then the graphs of the Atkinson index (computing EDE) and EDE-factors (computed using Definition 3.1) are graphed below. Our choice of graphing ϵ between 0 and 20 is deliberate as elicitation of ϵ in health or income generally falls in this range [4, 9, 14].





If inequality aversion is not considered, Scenario B results in a higher average income as four-fold more resources are additionally given compared to Scenario A. However, a computation of EDE income shows a more rapid decrease, with a tipping point at $\epsilon_{tip} \approx 3.37$ and $EDE(\epsilon_{tip}) \approx \$84,500$. This is also evident by the EDE-factors, as Scenario B puts a much higher weight on the lower income subgroup in EDE computations due to a larger income gap between subgroups.

Immediate properties from functional analysis

From our discussion in the previous Section, clearly the EDE-factors $w_i(\epsilon)$ are non-negative and sum to 1 for all ϵ:

$$\sum_i w_i(\epsilon) = 1.$$

At $\epsilon = 0$, this is simply the base case condition that the sum of all μ_i equal 1. These EDE-factors can be seen as a spiritual answer to questions raised in [10, 11] on a method to decompose measures, such as the Atkinson's index, as a simple weighted sum. We now discuss how EDE-factors generalize most of the homogeneity and transfer properties in Atkinson's index.

Income homogeneity. By Definition 3.1, the EDE-factors $w_i(\epsilon)$ are not affected by a uniform scaling of income levels $H_i \mapsto kH_i$ for some positive constant k. Thus, by Theorem 2.4, this implies income homogeneity for Atkinson's index, i.e. $EDE(\epsilon)$ is multiplied by the same constant k under a uniform scaling of income levels.

Population homogeneity. To show that the EDE-factors satisfy population homogeneity, suppose each subgroup i is replicated n times $(i,1),\dots,(i,n)$ and each replication is weighted $\omega_1,\dots,\omega_n$, with the ω_k's summing to 1. Then $H_{(i,1)} = \cdots = H_{(i,n)}$, and each subgroup (i,η) is weighted $\mu_i\omega_\eta$ in Atkinson's index. Therefore, its corresponding EDE-factor is

$$\begin{aligned} w_{(i,\eta)}(\epsilon) &= \left(\sum_\alpha \sum_j \left(\frac{H_{(j,\alpha)}}{H_{(i,\eta)}} \right)^{-\epsilon} \mu_j \omega_\alpha \right)^{-1} \mu_i \omega_\eta \\ &= \left(\sum_j \left(\frac{H_{(j,\alpha)}}{H_{(i,\eta)}} \right)^{-\epsilon} \mu_j \sum_\alpha \omega_\alpha \right)^{-1} \mu_i \omega_\eta \\ &= \omega_\eta \cdot w_i(\epsilon). \end{aligned}$$

As the weights $w_i(\epsilon)$ are q-analogs of the expression $f_{i,\epsilon}\mu_i$,

$$\begin{aligned} \left(\sum_\alpha \sum_i \frac{1}{H_{(i,\alpha)}} f_{(i,\alpha),\epsilon} \mu_i \omega_\alpha \right)^{-1} &= \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \sum_\alpha \omega_\alpha \right)^{-1} \\ &= \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1} \\ &= EDE(\epsilon) \end{aligned}$$

which proves population homogeneity.

Pigou-Dalton Principle. The Pigou-Dalton Principle is a transfer principle that asserts any social welfare function must prefer allocations that are more equitable. Formally, if $H_i > H_j$, then a transfer of $\Delta > 0$ from H_i to H_j, in such a way that

$$H_i - \Delta \geq H_j + \Delta^*, \qquad \Delta^* := \Delta \frac{\mu_i}{\mu_j}$$

must not decrease $EDE(\epsilon)$. This is easily seen to hold for the Atkinson index due to the concavity of the utility function.

We prove that the Pigou-Dalton Principle is a special case of Theorem 2.1 when $\epsilon > 2$, though the Theorem cannot be used to prove the Pigou-Dalton Principle at $0 < \epsilon < 2$. However, this is sufficient to show that Theorem 2.1 generalizes the Pigou-Dalton Principle for inequality studies using negative metrics (e.g. poverty level); see Section 5 for a discussion on this.

Proposition 3.3. *Theorem 2.1 implies the Pigou-Dalton Principle for Atkinson's index at $\epsilon > 2$.*

Proof. Let $\{H_k^*\}_k$ be the income profile such that $H_k^* = H_k$ for $k \neq i, j$, with $H_i^* = H_i - \Delta$ and $H_j^* = H_j + \Delta^*$ where Δ, Δ^* are as defined above. Let $EDE^*(\epsilon)$ be Atkinson's index calculated with the income profile $\{H_k^*\}_k$. By Theorem 2.1, there exists $f_{i,\epsilon}, f_{i,\epsilon}^*$ satisfying the conditions of that Theorem such that

$$EDE(\epsilon) = \left(\sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i \right)^{-1}$$

and

$$EDE^*(\epsilon) = \left(\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i \right)^{-1}.$$

We need to show that $EDE^*(\epsilon) \geq EDE(\epsilon)$. Note that

$$\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i = \Delta \mu_i \left(\frac{f_i^*}{H_i H_i^*} - \frac{f_j^*}{H_j H_j^*} \right) + \sum_i \frac{1}{H_i} f_{i,\epsilon}^* \mu_i.$$

The summation on the right satisfies

$$\sum_i \frac{1}{H_i} f_{i,\epsilon}^* \mu_i \leq \sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i$$

by Theorem 2.1. The term on the left satisfies

$$\begin{aligned} \frac{f_i^*}{H_i H_i^*} - \frac{f_j^*}{H_j H_j^*} &= EDE^*(\epsilon)^{\epsilon-2} \left(\frac{1}{H_i (H_i^*)^{\epsilon-1}} - \frac{1}{H_j (H_i^*)^{\epsilon-1}} \right) \\ &< 0 \end{aligned}$$

where the equality is by definition of f_i^* (Corollary 2.1) and the inequality is because $H_i > H_j$ and $H_i^* > H_j^*$. Therefore

$$\sum_i \frac{1}{H_i^*} f_{i,\epsilon}^* \mu_i \leq \sum_i \frac{1}{H_i} f_{i,\epsilon} \mu_i$$

implying $EDE^*(\epsilon) \geq EDE(\epsilon)$. □

A non-monotonic property

As $EDE(\epsilon)$ tends to the minimum income level as ϵ increases, the EDE-factors obey the following asymptotic property:

$$\lim_{\epsilon\to\infty} w_i(\epsilon) = \begin{cases} 1 & \text{if } H_i = \min\{H_k\}_k; \\ 0 & \text{otherwise.} \end{cases}$$

However, EDE-factors demonstrate a very interesting property: subgroups that do not correspond to the highest or lowest income may not be monotonically weighted as ϵ increases. More precisely, let $w_i(\epsilon)$ correspond to the EDE-factor of such a subgroup. By taking the derivative, one gets

$$\frac{d}{d\epsilon} w_i(\epsilon) = \left(\sum_j \left(\frac{H_j}{H_i}\right)^{-\epsilon} \mu_j\right)^{-2} \mu_i \left(\sum_j (\ln H_j - \ln H_i) \left(\frac{H_j}{H_i}\right)^{-\epsilon} \mu_j\right)$$

This is non-increasing exactly when the right-most sum is non-negative, or equivalently

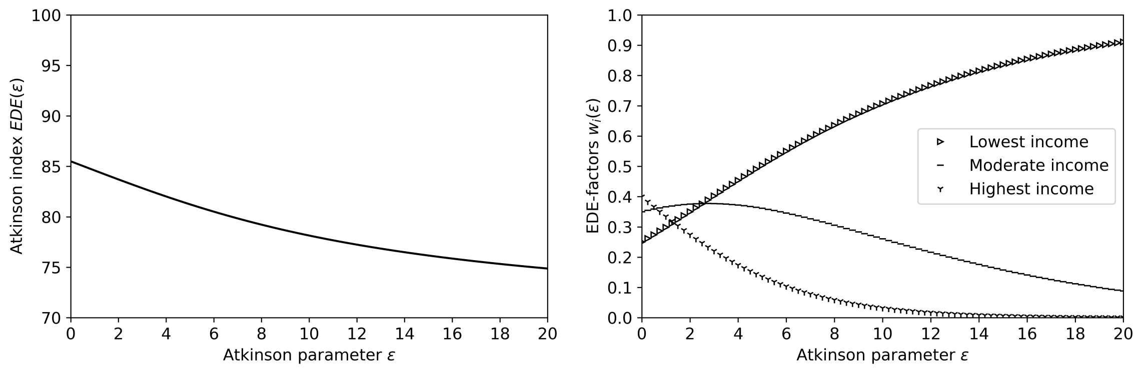
$$\ln H_i \geq \frac{\sum_{j\neq i} H_j^{-\epsilon} \mu_j \ln H_j}{\sum_{j\neq i} H_j^{-\epsilon} \mu_j}.$$

As the right hand side is a decreasing function in ϵ (by applying the Cauchy-Schwarz inequality on its derivative), this implies

$$\ln H_i \geq \sum_{j\neq i} \ln H_j^{\mu_j}.$$

If we assume $H_i > 1$ (with no loss of generality by income homogeneity), the above inequality implies its EDE-factor will increase at lower levels of ϵ to a unique maxima before monotonically tending to 0 as long as H_i is less than the relative geometric mean of the other subgroups. In other words, the Atkinson index may increasingly weight subgroups that are close to the lowest income for reasonably lower levels of ϵ. This fact cannot be seen directly from the original definition of the Atkinson index (Equation 1).

Example 3.4. Consider a population split into 3 subgroups with Atkinson weights $\mu_1 = 0.25$, $\mu_2 = 0.35$, $\mu_3 = 0.4$ and income levels $H_1 = \$70,000$ (lowest income), $H_2 = \$80,000$ (moderate income), $H_3 = \$100,000$ (highest income). Then the graphs of the Atkinson index and EDE-factors are graphed below.



In this example, the moderate income subgroup H_2 is increasingly weighted until $\epsilon_{2,peak} \approx 2.76$, with EDE-factor peaking at $w_2(\epsilon_{2,peak}) \approx 0.377$. This shows that at low levels of inequality aversion, the calculation of $EDE(\epsilon)$ puts emphasis in both H_1 and H_2, and not just the lowest income subgroup H_1.

4 Maximal EDE resource allocation

Consider the problem of reallocating current resources between subgroups to achieve maximal EDE income at inequality aversion ϵ. If p_i is the (local) production function of subgroup i and r_i is the amount of resources currently allocated to subgroup i, then the income level H_i can be expressed as $H_i = p_i r_i$. Assuming the sum of all current resources equals R, this reallocation problem reduces to the following optimization problem: Maximize

$$EDE(\epsilon) = \left(\sum_i (p_i r_i)^{1-\epsilon} \mu_i \right)^{\frac{1}{1-\epsilon}}$$

subject to the condition

$$\sum_i r_i = R.$$

If inequality aversion is not a consideration ($\epsilon = 0$), this problem has a simple solution: Allocate all resources to the subgroup with the highest value of $p_i \mu_i$, i.e. best weighted production function. However, if equity is a consideration ($\epsilon > 0$), we need to solve this problem via Lagrange Multipliers, telling us that $EDE(\epsilon)$ is maximized if $H_i =$

$\widetilde{H_{i,\epsilon}}$, where

$$\widetilde{H_{i,\epsilon}} := R\left(\sum_j \frac{1}{p_j}\left(\frac{p_j\mu_j}{p_i\mu_i}\right)^{\frac{1}{\epsilon}}\right)^{-1}. \tag{7}$$

Equation 7 can be compared to a similar setup described in [10].

EDE-factors offer a quick comparison of arriving at maximal EDE resource allocation given current allocation. Writing $w_{i,\epsilon} = w_i(\epsilon)$, Definition 3.1 implies

$$H_i = d\left(c_{i,\epsilon}^{-1}\mu_i\right)^{\frac{1}{\epsilon}}$$

for an expression d that is constant across all subgroups. As $H_i = p_i r_i$, dividing by p_i and summing across i gives

$$R = d\sum_j \frac{1}{p_j}\left(w_{j,\epsilon}^{-1}\mu_j\right)^{\frac{1}{\epsilon}},$$

and a rearrangement gives

$$H_i = R\left(\sum_j \frac{1}{p_j}\left(\frac{w_{j,\epsilon}^{-1}\mu_j}{w_{i,\epsilon}^{-1}\mu_i}\right)^{\frac{1}{\epsilon}}\right)^{-1} \tag{8}$$

which is very similar to Equation 7 for $\widetilde{H_{i,\epsilon}}$.

Equations 7 and 8 are useful for policy making as it allows us to compare resource allocation as a ratio between subgroups without requiring explicit knowledge on total resources (the R's cancel out under a ratio), allowing for scalability or if total resources are relatively unknown but with known effects. For instance, the expressions $w_{i,\epsilon}^{-1}\mu_i$ and $p_i\mu_i$, in the fractions of Equations 7 and 8 respectively, are related by a factor of $w_{i,\epsilon}p_i$:

$$p_i\mu_i = w_{i,\epsilon}p_i \cdot (w_{i,\epsilon}^{-1}\mu_i).$$

At current resource allocation, this means that, for each subgroup i, the product $w_{i,\epsilon}p_i$ consisting of EDE-factor ($w_{i,\epsilon}$) and production function (p_i) can serve as a measure of "farness" compared to the optimal resource allocation that gives rise to the maximal EDE at a certain $\epsilon > 0$.

5 EDE calculations for negative metrics

Let M be a metric that varies inversely proportional to income level H (e.g. poverty level, death rate). Such metrics are important in applications for both economics and health [5]. Note that the EDE-adjusted M as ϵ varies cannot be calculated by directly substituting M into Atkinson's index $EDE(\epsilon)$ (Equation 1) as this would tend to the lowest level of M (i.e. highest level of income), contrary to what we expect.

With that said, calculations on negative metrics M can be done through our functional equation (Theorem 2.5). For EDE-adjusted calculation on M, we require tending to the highest level (i.e. lowest level of income) as ϵ increases. Furthermore, we would like larger values of subgroup-level M to be emphasized so that the most disadvantageous subgroups bear more magnitude in the computation of an EDE-adjusted M. Therefore, we desire an arithmetic-like sum

$$EDE^{\dagger}(\epsilon) := \sum_i M_i g_{i,\epsilon} \mu_i,$$

where $g_{i,\epsilon}$ are functions depending on M_i and ϵ. This expression must be consistent with the usual arithmetic sum without any considerations on inequality aversion, i.e.

$$EDE^{\dagger}(0) = \sum_i M_i \mu_i.$$

Therefore, the arithmetic-like sum we seek is the expression $L(M_i^{-1}, 2+\epsilon)$ in Theorem 2.5 as $f_{i,2+\epsilon} = 1$ by Corollary 2.2. By the same Theorem

$$EDE^{\dagger}(\epsilon) = L(M_i^{-1}, 2+\epsilon) = L(M_i, -\epsilon)^{-1} = EDE(-\epsilon). \tag{9}$$

Equation 9 is a generalization of Sterck's observation [16] that minimizing overall poverty level is equivalent to maxizing EDE income at $\epsilon = 2$, for this observation is simply a consequence of substituting $\epsilon = 0$ into the our Equation. In other words, our Equation extends a pointwise equivalent into a structural equivalence valid for all $\epsilon \geq 0$. This is summarized as the Duality Principle below.

Duality Principle 5.1. Minimizing the EDE of a negative metric M at ϵ is equivalent to maximizing the EDE of its inverse metric at $\epsilon + 2$.

Everything discussed in Sections 3 and 4 can be appropriately carried over to negative metrics by replacing ϵ with $-\epsilon$.

EDE-factors for negative metrics

For negative metrics, the EDE-factors are

$$w_i^\dagger(\epsilon) := \left(\sum_j \left(\frac{M_i}{M_j} \right)^{-\epsilon} \mu_j \right)^{-1} \mu_i = \left(\frac{M_i}{EDE(-\epsilon+1)} \right)^{\epsilon} \mu_i.$$

The three main properties still hold (income homogeneity, population homogeneity, Pigou-Dalton Principle). In fact, our functional-analytic discussion in Section 2 is actually a generalization of PDP in this case.

Corollary 5.2. *Theorem 2.1 implies the Pigou-Dalton Principle for negative metrics at all* $\epsilon \geq 0$.

Proof. This is immediate by applying Proposition 3.3 to Equation 9. □

The non-monotonic property of EDE-factors works the opposite way for negative metrics: $w_i^\dagger(\epsilon)$ is strictly non-increasing as ϵ increases precisely when

$$\ln M_i \leq \sum_{j \neq i} \ln M_j^{\mu_j}.$$

Resource allocation for negative metrics

Let M be a negative metric. Typically, negative metrics are rates or probabilities (such as poverty level), and resource allocation problems seek to optimally allocate an amount of new resources in order to lower M. This is an important area of research in cost-effectiveness analysis, and while EDE-factors cannot globally solve the issue of resource allocation to minimize M, it can offer a measurement on how far a proposed allocation strategy is from a hypothetical scenario where both M and resources can be traded to achieve the minimal EDE M as ϵ varies.

We outline the modifications required to apply techniques in Section 4. Let R be the total amount of new resource to be allocated, and let M_i, p_i, r_i be the respective negative metric, production function, and amount of resources allocated to subgroup i. If M_i is transformed to $M_i^o = M_i - p_i r_i$ after resource reallocation, and $w_i^o(\epsilon) = w_{i,\epsilon}^o$

is the respective EDE-factor after resource allocation, then

$$M_i^o = \left(-R + \sum_j \frac{M_j}{p_j}\right)\left(\sum_j \frac{1}{p_j}\left(\frac{(w_{i,\epsilon}^o)^{-1}\mu_i}{(w_{j,\epsilon}^o)^{-1}\mu_j}\right)^{\frac{1}{\epsilon}}\right)^{-1}.$$

If we allow a hypothetical scenario, where the minimal of

$$EDE^o(\epsilon) = \left(\sum_i (M_i - p_i r_i)^{1-\epsilon}\mu_i\right)^{\frac{1}{1-\epsilon}}$$

can be attained subject to the condition

$$\sum_i r_i = R,$$

then Lagrange Multipliers imply this can be achieved when $M_i = \widetilde{M_{i,\epsilon}}$, where

$$\widetilde{M_{i,\epsilon}} := \left(-R + \sum_j \frac{M_j}{p_j}\right)\left(\sum_j \frac{1}{p_j}\left(\frac{p_i\mu_i}{p_j\mu_j}\right)^{\frac{1}{\epsilon}}\right)^{-1}.$$

Notice this scenario where $M_i = \widetilde{M_{i,\epsilon}}$ is necessarily hypothetical as $\widetilde{M_{i,\epsilon}}$ may be larger than M_i. In the context of poverty level, this means we are removing enough wealth from a subgroup to cause more people to live in poverty, which is not a realistic scenario. However, $\widetilde{M_{i,\epsilon}}$ can serve as a benchmark on how far current resource allocation is to achieving the lowest EDE M at a certain inequality level $\epsilon > 0$.

6 Concluding remarks

This paper demonstrated a new decomposition of the Atkinson index by way of EDE-factors (Equation 3.1). Although many kinds of income inequality measures exist [3], we chose to focus on the Atkinson index for applicability in both health and economics. In health, studies have shown the Atkinson index may be the most appropriate index for inequality analyses [7] as it allows for many different interpretations of subgroup decomposability, satisfies the Pigou-Dalton Principle, and avoid value judgement. In economics, Shorrocks [11, 12] gave mathematical justification for the generalized entropy index to be the family of inequality measures for our purposes, and this index can be viewed as a monotonic transformation of the Atkinson index.

The technical aspect of our paper contributes a novel way to decompose the Atkinson index as we used an approach via a number-theoretic viewpoint. Our main objectives were to seek a general principle behind Atkinson's observed duality between income at $\epsilon = 2$ and poverty at $\epsilon = 0$, as well as a broadening of the Pigou-Dalton Principle. A number-theoretic viewpoint is essential to obtain our duality principle as it is mathematically expressed via a functional equation (Theorem 2.5). This duality is also hinted at in current working papers on decomposition of measures [8, 16]. As for the broadening of the Pigou-Dalton Principle, our result (Proposition 5.2; Corollary 5.2) does not require any reference on the direction of income allocation.

We believe the framework developed in this paper may be generalized to more classes of income inequality measures satisfying the three conditions listed at the start of this paper (subgroup decomposability; Pigou-Dalton principle; avoids value judgement). In particular, for inequality metrics that avoid value judgement by introducing an explicit parameter ϵ, a functional equation (dependent on ϵ) for a number-theoretic-like L-function arising from subgroup decomposability should imply a duality principle, while an analog of the Riesz-Fréchet Representation Theorem should broaden on the Pigou-Dalton Principle.

References

[1] Atkinson, A. B. (1970). *On the measurement of inequality.* Journal of Economic Theory, 2(3), 244–263.

[2] Bullen, P. S. (2003). *Handbook of means and their inequalities (2nd ed.).* Springer.

[3] De Maio, F. G. (2007). *Income inequality measures.* Journal of Epidemiology & Community Health, 61(10), 849–852.

[4] Hurley, J., Mentzakis, E., Walli-Attaei, M. (2020). *Inequality aversion in income, health, and income-related health.* Journal of Health Economics, 70, 102276.

[5] Kim, H.-Y., Bershteyn, A., Russo, R., Mcgillen, J., Sisti, J., Ko, C., Shaff, J., Newton-Dame, R., Braithwaite, R. S. (2025). *Does prioritization of COVID vac-*

cine distribution to communities with the highest COVID burden reduce health inequity? Journal of Infection and Public Health, 18(11), 102904.

[6] Krantz, S. G., Parks, H. R. (2002). *A primer of real analytic functions (2nd ed.).* Birkhäuser.

[7] Levy, J. I., Chemerynski, S. M., Tuchmann, J. L. (2006). *Incorporating concepts of inequality and inequity into health benefits analysis.* International Journal for Equity in Health, 5(1), 2.

[8] Moramarco, D., Sterck, O. (2025). *Who drives inequality?* SSRN working paper, 5220560.

[9] Robson, M., Asaria, M., Cookson, R., Tsuchiya, A., Ali, S. (2016). *Eliciting the level of health inequality aversion in England.* Health Economics, 26(10), 1328–1334.

[10] Robson, M., O'Donnell, O., Tom Van Ourti. (2024). *Aversion to health inequality – Pure, income-related and income-caused.* Journal of Health Economics, 94, 102856.

[11] Shorrocks, A. F. (1980). *The class of additively decomposable inequality measures.* Econometrica, 48, 613-625.

[12] Shorrocks, A. F. (1984). *Inequality decomposition by population subgroups.* Econometrica, 52(6), 1369.

[13] Shorrocks, A. F. (2012). *Decomposition procedures for distributional analysis: a unified framework based on the Shapley value.* The Journal of Economic Inequality, 11(1), 99–126.

[14] Slejko, J. F., Ricci, S., dosReis, S., Cookson, R., Kowal, S. (2025). *Health inequality aversion in the United States.* Value in Health, 29(1), 129–138.

[15] Stein, E. M., Shakarchi, R. (2011). *Functional analysis: Introduction to further topics in analysis (Princeton Lectures in Analysis, Vol. 4).* Princeton University Press.

[16] Sterck, O. (2024). *Poverty without poverty line.* SSRN working paper, 4785458.